Kronecker product $\otimes$

- Given $B \in \mathbb{R}^{m \times n}$ and $a \in \mathbb{R}$, aB denotes matrix in which every entry of B is multiplied by a:

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} \quad aB = \begin{bmatrix} ab_{11} & \cdots & ab_{1n} \\ \vdots & & \vdots \\ ab_{m1} & \cdots & ab_{mn} \end{bmatrix}$$

- a block matrix A is a matrix that has been organized into blocks (submatrices)
- write $A = (A_{ij})$ if A_{ij} is ij-th block
- blocks don't have to be same size, but blocks on same row must have same # of rows, blocks on same column same # of columns
- We say $A = (A_{ij})$ is "block $m \times n$" if it has $m \times n$-many blocks, i.e. i ranges $1 \le i \le m$, j ranges $1 \le j \le n$.

$$A_{ij} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$$

"block $m \times n$"

e.g.

if $A = \left[\begin{array}{c|cc} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$

is block 2×2

- If $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ then the Kronecker product $A \otimes B$ is the block $m \times n$ matrix where ij-th block is $a_{ij} B$ (so $A \otimes B \in \mathbb{R}^{mp \times nq}$)

if $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$

then $A \otimes B = \begin{bmatrix} a_{11} B & \cdots & a_{1n} B \\ a_{m1} B & \cdots & a_{mn} B \end{bmatrix}$

e.g. if $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

then $A \otimes B = \begin{bmatrix} 1B & 2B \\ -1B & 0B \end{bmatrix}$

$$= \left[\begin{array}{cc|cc} 1 & 2 & 2 & 4 \\ 3 & 4 & 6 & 8 \\ \hline -1 & -2 & 0 & 0 \\ -3 & -4 & 0 & 0 \end{array}\right]$$

2x2 block
4x4 overall.

Properties of $\otimes$

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
- $(A \otimes B)(C \otimes D) = AC \otimes BD$ "mixed product property"

usual matrix product

assuming product defined (dim's make sense)

- $(A \otimes B)^T = A^T \otimes B^T$
- $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ (assuming A, B square invertible)

using them, can prove:

orthogonal $\otimes$ orthogonal = orthogonal
sym. pos. def $\otimes$ sym. pos. def = sym. pos. def

If A, B square:
eigenvalues of $A \otimes B$ = all ~~products~~ products
$\alpha_i \beta_j$ where α_i eigenvalue of A, β_j eigenvalue of B

In general:
singular values of $A \otimes B$ = " "
$\alpha_i \beta_j$ " singular value ", " singular value of B.

<u>more</u>: $\text{rank}(A \otimes B) = \text{rank}(A)\,\text{rank}(B)$

$\|A \otimes B\| = \|A\|\,\|B\|$

where $\|\cdot\|$ could be 2-norm or F-norm

<u>Khatri – Rao product $\odot$</u>

– if $A = (A_{ij})$ $B = (B_{ij})$ are block $m \times n$ matrices, the <u>Khatri – Rao</u> product $A \odot B$ is the $m \times n$ block matrix whose ijth block is $A_{ij} \otimes B_{ij}$ (so: KR product only defined relative to some blocking of the matrices A, B

e.g. if

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}$$

then

$$A \odot B = \begin{bmatrix} A_{11} \otimes B_{11} & A_{12} \otimes B_{12} \\ A_{21} \otimes B_{21} & A_{22} \otimes B_{22} \\ A_{31} \otimes B_{31} & A_{32} \otimes B_{32} \end{bmatrix}$$

We'll care about Khatri-Rao when blocks are the columns of a given matrix:

$$[\vec{a}_1 | \cdots | \vec{a}_n] \odot [\vec{b}_1 | \cdots | \vec{b}_n]$$

$$= [\vec{a}_1 \otimes \vec{b}_1 | \cdots | \vec{a}_n \otimes \vec{b}_n]$$

e.g.

$$\left[\begin{array}{c|c} 1 & 3 \\ 2 & 4 \end{array}\right] \odot \left[\begin{array}{c|c} 1 & 2 \\ -1 & 0 \end{array}\right]$$

$$= \left[\begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \;\middle|\; \begin{bmatrix} 3 \\ 4 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right]$$

$$= \left[\begin{array}{c|c} 1 & 6 \\ -1 & 0 \\ 2 & 8 \\ -2 & 0 \end{array}\right]$$

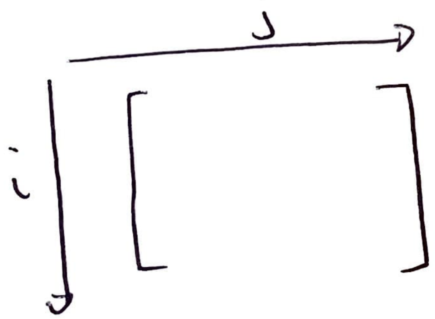
Tensors

- An <u>order-d</u> <u>tensor</u> $T \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$ is a d-dimensional array of (real) numbers.
    - ↳ order 1 tensors are vectors (columns of data)
    - ↳ order 2 tensors are matrices (rectangles of data)
    - ↳ order 3 tensors are "boxes of data"

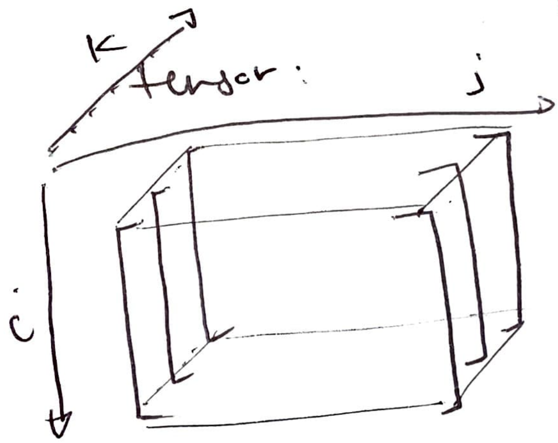
    ⋮ etc.

- we will focus on order 3 tensors, though everything generalizes
- in an order 3 tensor $T \in \mathbb{R}^{I \times J \times K}$, entries are determined by three coordinates: $i \le I$, $j \le J$, $k \le K$

matrix:



tensor:



Unfoldings

- hard to draw order 3 tensors (and impossible for higher orders), so we want to "unfold" them into matrices

- many ways to do this — we isolate 3: the "modal unfoldings" of T

- first: the modal fibers of $T \in \mathbb{R}^{I \times J \times K}$ are the vectors obtained by fixing j, k and letting i vary $1 \leq i \leq I$.

  " mode 2 fibers " "
  " " fixing i, k "
  " j vary $1 \leq j \leq J$
  " mode 3 fibers " "
  " fixing i, j
  " k vary $1 \leq k \leq K$.

- the l-th modal ($l \in \{1,2,3\}$) unfolding T_l is the matrix whose columns are the mode l fibers.