Summary of LS:

- To minimize \( \|A\tilde{x} - b\| \) for "tall-skinny" A, use \( \tilde{x} = A^+ b \) (least squares).

- To minimize \( \|Ax - b\|_F \), just minimize \( \|A_{i\cdot} - b_i\|_F \) individually, where \( \tilde{e}_i \)'s, \( b_i \)'s columns of \( x, b \) respectively ("matrix least squares").

Back to CP: Given \( T \) of rank \( r \) and \( \tilde{z} \) \& \( \tilde{v} \)

Want to find \( \tilde{u}_w, \tilde{v}_w, \tilde{z}_w \) minimizing

\[
\|T - \sum_{i=1}^{k} \tilde{q}_i \tilde{v}_i \tilde{u}_i \|_F = \|T_1 - U (W_{0W})^T \|_F
\]

\[
= \|T_2 - V (W_{0U})^T \|_F
\]

\[
= \|T_3 - W (V_{0Y})^T \|_F
\]

\( \tilde{u}_w, \tilde{v}_w, \tilde{z}_w \) columns of \( W_{0W} \).

\( r^2 \) LS problems - 3 unknown matrices, \( r^2 - 1 \).

- Take iterative approach:

  1st stage: pick arbitrary matrices \( V, W \) and minimize \( \|T_1 - U (W_{0V})^T \|_F \) to find a metric \( U \).
- This is (matrix) least squares of the form

\[
\min_x \| B - X A \|_F
\]

Note: Since unknown on left we'll need to transpose to get same form as above.

\[
\| B - X A \|_F = \| X A - B \|_F = \| (X A - B)^T \|_F
\]

\[
= \| (X A)^T - B^T \|_F
\]

\[
= \| A^T X - B^T \|_F
\]

\[
= \| A^T X - B^T \|_F
\]

\[
\Rightarrow \text{same form as before.}
\]

2nd stage: Keeping same \( w \), minimize

\[
\| T_2 - V (w_0 u_0) \|_F
\]

to find \( u_1 \).

3rd: Minimize \n\[
\| T_3 - W (V, u_0) \|_F
\]

to find \( u_1 \).

In general: \( u_1, v_1, w \) will only approximate a solution to minimization problem above. So we may continue...
- Minimize $\| T_1 - U (W_1, W_2) \|_F$
  to find $U_2$

- Minimize $\| T_2 - V (W_1, U_2) \|_F$
  to find $V_2$

- Minimize $\| T_3 - W (V_2, U_2) \|_F$
  to find $W_2$

The sequence $U_1 \rightarrow U_2 \rightarrow \ldots$
$V_1 \rightarrow V_2 \rightarrow \ldots$
$W_1 \rightarrow W_2 \rightarrow \ldots$

will (hopefully) converge to an actual solution, though I don't know if there are good estimates for how long this will take in a given instance.

- Once we've gone "far enough"
(found some $U_n = [u_1, \ldots, u_k]$
$V_n = [v_1, \ldots, v_k]$
$W_n = [w_1, \ldots, w_k]$)
we can unpack columns to get our sum of rank 1 tensors:
\[
\sum_{k \in \mathbb{R}} u_k v_k \omega_k^T
\]
which, approximately, minimizes \( \| T - \sum_{k \in \mathbb{R}} u_k v_k \omega_k^T \|_F \)
over all such \( k \) length sums.

\[ #5 \] on \#5 is first step in such approx. process.

\[ T = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \rightarrow T_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \]

\( \text{rank 2 (why?)} \)

Want rank 1 tensor \([u_2^1 v_2^1 w_2^1]\).

Closest to \( T \) if we assume \([v_2^1 w_2^1] = [1]\).

i.e., want to solve: \( \min \| T_1 - \tilde{u} (\tilde{v} \tilde{w})^T \|_F \)

\( \min \| T_1 - \tilde{u} (\tilde{v} \tilde{w})^T \|_F \)

\( \min \| T_1 - \begin{bmatrix} u_2^1 v_2^1 w_2^1 \end{bmatrix} ([1])^T \|_F \)

\( \min \| [u_2^1 v_2^1] c_{111}^1 - T_1 \|_F \)
of form "$11xA = 5111$" we want "$11Ax = 5111$" \( ^V \)

So we transpose:

$$
\min \| A [ \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} ] \|_2
$$

(In this case our column vectors $\bar{x}_1, \bar{x}_2$
are just scalars $u_1, u_2$)

this splits into two $L^2$ problems:

$$
\min \| [ \begin{pmatrix} 1 \\ 0 \end{pmatrix} ] u_1 - [ \begin{pmatrix} 1 \\ 0 \end{pmatrix} ] \|_2
$$

$$
\min \| [ \begin{pmatrix} 0 \\ 1 \end{pmatrix} ] u_2 - [ \begin{pmatrix} 0 \\ 1 \end{pmatrix} ] \|_2
$$

the second is easy: solved by $u_2 = 1$
we could use calculus to solve the first, which reduces to

$$
\minimize: \ u_1^2 + (u_1 - 1)^2 + (u_1 - 1)^2 + (u_1 - 0)^2
$$

but lets instead solve by least squares—
need to find pseudo-inverse for $[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]$

As first need its SVD (finding SVD of a vector Oh!)
\[
\begin{bmatrix}
\hat{v}
\end{bmatrix}
= 
\begin{bmatrix}
\cdot \\
\cdot \\
\cdot 
\end{bmatrix}
= \sqrt{v}
\begin{bmatrix}
\frac{v}{\sqrt{v}} \\
\frac{v}{\sqrt{v}} \\
\frac{v}{\sqrt{v}}
\end{bmatrix}
\]

what unit "vector" stretched most by \( \hat{v} \) - only one choice (up to sign)!

\[
\Sigma
\begin{bmatrix}
\cdot \\
\cdot \\
\cdot 
\end{bmatrix}
= 
\begin{bmatrix}
\frac{v}{\sqrt{v}} \\
\frac{v}{\sqrt{v}} \\
\frac{v}{\sqrt{v}}
\end{bmatrix}
\begin{bmatrix}
\sqrt{v} \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
\cdot \\
\cdot \\
\cdot 
\end{bmatrix}
\]

\[
\begin{bmatrix}
\cdot \\
\cdot \\
\cdot 
\end{bmatrix}
= 
\begin{bmatrix}
\cdot \\
\cdot \\
\cdot 
\end{bmatrix}
\]

So \( A^+ = V \Sigma^+ U^T \)

\[
= [1] \begin{bmatrix}
\frac{v}{\sqrt{v}} \\
\frac{v}{\sqrt{v}} \\
\frac{v}{\sqrt{v}} \\
\frac{v}{\sqrt{v}}
\end{bmatrix}
\]

Want to minimize \( \| A^T \hat{u} \| - [b] \|

So \( \hat{u} = \frac{A^+ b}{v} = [1] \begin{bmatrix}
\frac{v}{\sqrt{v}} \\
\frac{v}{\sqrt{v}} \\
\frac{v}{\sqrt{v}} \\
\frac{v}{\sqrt{v}}
\end{bmatrix}
\begin{bmatrix}
\frac{v}{\sqrt{v}} \\
\frac{v}{\sqrt{v}} \\
\frac{v}{\sqrt{v}} \\
\frac{v}{\sqrt{v}}
\end{bmatrix}
\]

\[
= [\frac{v}{\sqrt{v}} \cdot \cdot \cdot \frac{v}{\sqrt{v}}] \begin{bmatrix}
\frac{v}{\sqrt{v}} \\
\frac{v}{\sqrt{v}} \\
\frac{v}{\sqrt{v}} \\
\frac{v}{\sqrt{v}}
\end{bmatrix}
\]

\[
= \frac{3}{4} \hat{u}
\]
So: \( \| {u'} \|_{H} - \| v \|_{H} \) is minimized when \( u_{1} = \frac{3}{4} \).

Bringing it all back: if \( [v_{1}] = [v_{2}] = [w_{2}] = [i] \) then \( \alpha_{1} [u_{1}'] \circ [v_{1}] \circ [v_{2}] \circ [u_{2}] = (\tilde{u}_1 (\tilde{w}' \tilde{v})) \) when unfolded.

\( u \) closest to \( 0 \) to \( \Gamma \) when \( u_{1} = \frac{3}{4} \) and \( u_{2} = 1 \).

Let: \( [i] \circ [i] \circ [i] = \alpha_{1} \)

\( \tilde{u}_{1} = \tilde{v}_{1} = \tilde{w}_{1} \) such rank 1 target does \( \Gamma \).