Summary of LS:

- to minimize $\|A\vec{x}-\vec{b}\|$ for "tall-skinny" A let $\hat{x} = A^{+}\vec{b}$ (least squares)

- to minimize $\|AX - B\|_F$ just minimize $\|A\vec{x}_1-\vec{b}_1\|, \ldots, \|A\vec{x}_n-\vec{b}_n\|$ individually, where $\vec{x}_i$'s, $\vec{b}_i$'s columns of X, B respectively ("matrix least squares")

Back to CP: Given T of rank r and $k < r$ want to find $\vec{u}_i$'s, $\vec{v}_i$'s, $\vec{w}_i$'s minimizing

$$\|T - \sum_{i=1}^{k} \vec{u}_i \circ \vec{v}_i \circ \vec{w}_i\|_F = \|T_1 - U(W \odot V)^T\|_F$$
$$= \|T_2 - V(W \odot U)^T\|_F$$
$$= \|T_3 - W(V \odot U)^T\|_F$$

$\vec{u}_i, \vec{v}_i, \vec{w}_i$ columns of U, V, W.

not LS problems — 3 unknown matrices, not 1.

- take iterative approach:

1st stage: pick arbitrary matrices V, W and minimize $\|T_1 - U(W \odot V)^T\|_F$ to find a matrix U_1

- thus $\underline{\underline{W}}$ (matrix) least squares: of the form $\min_X \|B - XA\|_F$ (ii)

note: ↑ since unknown on left we'll need to transpose to get same form as above

$$\|B - XA\|_F = \|XA - B\|_F \quad \text{(negative cancels)}$$
$$= \|(XA - B)^T\|_F \quad \text{(same entries)}$$
$$= \|(XA)^T - B^T\|_F$$
$$= \|A^T X^T - B^T\|_F$$
$$= \|A' X' - B'\|_F$$

↳ same form as before.

2nd stage: keeping same W, minimize $\|T_2 - V(W \odot U_1)^T\|_F$ to find V_1.

3rd: minimize $\|T_3 - W(V_1 \odot U_1)^T\|_F$ to find W_1.

in general: U_1, V_1, W_1 will only approximate a sol'n to minimization problems above. So we may continue...

(iii)

- minimize $\|T_1 - U(W_1 \odot V_1)^T\|_F$
  to find U_2

"Alternating LS"

- minimize $\|T_2 - V(W_1 \odot U_2)^T\|_F$
  to find V_2

- minimize $\|T_3 - W(V_2 \odot U_2)^T\|_F$
  to find W_2

the sequences $U_1 \to U_2 \to \dots$
$V_1 \to V_2 \to \dots$
$W_1 \to W_2 \to \dots$

will (hopefully) converge to an actual solution, though I don't know if there are good estimates for how long this will take in a given instance.

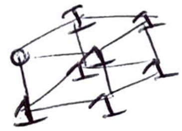
- Once we've gone "far enough"
(found some $U_n = [\vec{u}_1 | \dots | \vec{u}_K]$
$V_n = [\vec{v}_1 | \dots | \vec{v}_K]$
$W_n = [\vec{w}_1 | \dots | \vec{w}_K]$)

we can unpack columns to get our sum of rank 1 tensors:

$$\sum_{c=1}^{k} \vec{u}_c \circ \vec{v}_c \circ \vec{w}_c$$

which, approximately, minimizes $\|T - \sum_{c=1}^{k} \vec{u}_c \circ \vec{v}_c \circ \vec{v}_c\|_F$ over all such K length sums.

~~????~~ #5 on HW is first step in such an approx'n process.

T =  $\longrightarrow T_1 = \left[\begin{matrix}0 & 1\\ 1 & 1\end{matrix} \,\middle|\, \begin{matrix}1 & 1\\ 1 & 1\end{matrix}\right]$

is rank 2 (why?)

<u>Want</u> rank 1 tensor $\overset{\vec{u}}{\begin{bmatrix}u_1\\u_2\end{bmatrix}} \circ \overset{\vec{v}}{\begin{bmatrix}v_1\\v_2\end{bmatrix}} \circ \overset{\vec{w}}{\begin{bmatrix}w_1\\w_2\end{bmatrix}}$

closest to T, if we assume $\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}w_1\\w_2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

i.e. want to solve:

$$\min_{\vec{u}} \|T_1 - \underset{\vec{u}}{u}(\underset{\vec{w}}{w} \circ \underset{\vec{v}}{v})^T\|_F$$

i.e. $$\min_{\vec{u}} \|T_1 - \vec{u}(\vec{w} \otimes \vec{v})^T\|_F$$

$$\min_{u} \|T_1 - \begin{bmatrix}u_1\\u_2\end{bmatrix}\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)^T\|_F$$

$$\min_{\vec{u}} \|\begin{bmatrix}u_1\\u_2\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} - T_1\|_F$$

of form "$\|XA - B\|$" we want "$\|AX - B\|$"

So we transpose:

$$\min \left\| \overset{A}{\underset{A}{\begin{bmatrix} | \\ | \\ | \end{bmatrix}}} \overset{X}{\underset{\vec{x}_1 \quad \vec{x}_2}{[u_1, u_2]}} - \overset{B}{\underset{\vec{b}_1 \quad \vec{b}_2}{\begin{bmatrix} 0 & | \\ | & | \\ | & | \end{bmatrix}}} \right\|_F$$

(in this case our column vectors $\vec{x}_1, \vec{x}_2$ are just scalars u_1, u_2)

this splits into two LS problems

$$\min \left\| \begin{bmatrix} | \\ | \\ | \end{bmatrix} u_1 - \begin{bmatrix} 0 \\ | \\ | \end{bmatrix} \right\|$$

$$\min \left\| \begin{bmatrix} | \\ | \\ | \end{bmatrix} u_2 - \begin{bmatrix} | \\ | \\ | \end{bmatrix} \right\|$$

the second is easy: solved by $u_2 = 1$.

we could use calculus to solve the first, which reduces to

minimize: $u_1^2 + (u_1 - 1)^2 + (u_1 - 1)^2 + (u_1 - 1)^2$

but let's instead solve by least squares — need to find pseudo-inverse for $\begin{bmatrix} | \\ | \end{bmatrix}$

↳ first need its SVD! finding SVD of a vector... ok!

(vi)

$$\overset{A}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}[1] = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \sqrt{4}\begin{bmatrix}1/\sqrt{4}\\1/\sqrt{4}\\1/\sqrt{4}\\1/\sqrt{4}\end{bmatrix}$$

what unit "vector" stretched most by $[1]$ – only one choice (up to sign)!

doesn't matter, just extend to orthonormal basis

$$\text{So}\quad \underset{4\times 1}{\overset{A}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}} = \underset{U}{\begin{bmatrix}1/\sqrt{4} & \boxed{}\\1/\sqrt{4} & \\1/\sqrt{4} & \\1/\sqrt{4} & \end{bmatrix}}\underset{\Sigma}{\begin{bmatrix}\sqrt{4}\\0\\0\\0\end{bmatrix}}\underset{V^T}{[1]}$$

$$\text{So}\quad A^+ = V\Sigma^+U^T$$

$$= [1]\,[1/\sqrt{4}\;0\;0\;0]\begin{bmatrix}1/\sqrt{4} & 1/\sqrt{4} & 1/\sqrt{4} & 1/\sqrt{4}\\ & \boxed{} & & \end{bmatrix}$$

Want to minimize $\left\| \overset{A}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}\overset{x}{x_1} - \overset{\vec{b}}{\begin{bmatrix}0\\1\\1\\1\end{bmatrix}} \right\|$

Sol'n is $\hat{x} = A^+\vec{b} = [1]\,[1/\sqrt{4}\;0\;0\;0]\begin{bmatrix}1/\sqrt{4} & 1/\sqrt{4} & 1/\sqrt{4} & 1/\sqrt{4}\\ & \boxed{} & & \end{bmatrix}\begin{bmatrix}0\\1\\1\\1\end{bmatrix}$

$$= (1/\sqrt{4}\;0\;0\;0)\begin{bmatrix}3/\sqrt{4}\\0\end{bmatrix}$$

$$= 3/4$$

So: $\left\| \begin{bmatrix}1\\1\\1\\1\end{bmatrix}u_1 - \begin{bmatrix}0\\1\\1\\1\end{bmatrix} \right\|$ is minimized

when $u_1 = 3/4$

bringing it all back: if $\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}w_1\\w_2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

then $\underset{\vec{u}}{\begin{bmatrix}u_1\\u_2\end{bmatrix}} \circ \underset{\vec{v}}{\begin{bmatrix}v_1\\v_2\end{bmatrix}} \circ \underset{\vec{w}}{\begin{bmatrix}w_1\\w_2\end{bmatrix}}$ ($\vec{u}(\vec{w} \otimes \vec{v})$ when unfolded)

is closest to to T when $u_1 = 3/4$, $u_2 = 1$

i.e. $\begin{bmatrix}3/4\\1\end{bmatrix} \circ \begin{bmatrix}1\\1\end{bmatrix} \circ \begin{bmatrix}1\\1\end{bmatrix} =$

is closest such rank 1 tensor to T =