1. (Strang I.7.23) Suppose that $C$ is symmetric and positive definite (so $x^TCx > 0$ for all $x \neq 0$) and $A$ has independent columns. Check that the matrix $S = A^TCA$ is also positive definite.

2. (Strang I.7.28) Suppose that $S$ is symmetric positive definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$.

   a) What are the eigenvalues of the matrix $\lambda_1 I - S$? Conclude that this matrix is positive semi-definite.

   b) It follows that $\lambda_1 x^T x \geq x^T S x$ for all $x$. Why? Conclude that the maximum value of the function $f(x) = \frac{x^T S x}{x^T x}$ is $\lambda_1$.

   (Equivalently, this gives that the maximum value of the quadratic form $P(x) = x^T S x$ subject to the constraint $x^T x = 1$ is $\lambda_1$.)

3. (Strang I.7.21) Draw the tilted ellipse $x^2 + xy + y^2 = 1$ and find the half-lengths of its axes from the eigenvalues of the corresponding symmetric matrix $S$.

4. (Strang I.9.2) Find a closest rank-1 approximation to the following matrices:

   \[
   \begin{bmatrix}
   3 & 0 & 0 \\
   0 & 2 & 0 \\
   0 & 0 & 1
   \end{bmatrix},
   \begin{bmatrix}
   0 & 3 \\
   2 & 0
   \end{bmatrix},
   \begin{bmatrix}
   1 & \frac{1}{2} \\
   \frac{1}{2} & 1
   \end{bmatrix},
   \begin{bmatrix}
   \cos(\theta) & -\sin(\theta) \\
   \sin(\theta) & \cos(\theta)
   \end{bmatrix}
   \]

   Following up on the third matrix: what are the singular values of a symmetric matrix, in general? On the fourth: what about of an orthogonal matrix?

5. (Strang I.12.10) Suppose all entries are 1 in a $2 \times 2 \times 2$ tensor $T$, except the first entry $t_{111} = 0$. Write $T$ as a sum of two rank-1 tensors.

   Solve the following optimization problem:

   Find the closest rank-1 tensor $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \circ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \circ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $T$ (in the Frobenius norm), assuming that $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   (This is a least squares problem – can you see how to frame it that way?)

6. (Strang I.12.8) The largest possible rank of a $2 \times 2 \times 2$ tensor is 3. Can you find an example?