Homework #4

1. (Strang I.7.23) Suppose that C is symmetric and positive definite (so $\mathbf{x}^T C\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$) and A has independent columns. Check that the matrix $S = A^T CA$ is also positive definite.

2. (Strang I.7.28) Suppose that S is symmetric positive definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$.
   a) What are the eigenvalues of the matrix $\lambda_1 I - S$? Conclude that this matrix is positive semi-definite.
   b) It follows that $\lambda_1 \mathbf{x}^T\mathbf{x} \geq \mathbf{x}^T S\mathbf{x}$ for all $\mathbf{x}$. Why? Conclude that the maximum value of the function $f(\mathbf{x}) = \frac{\mathbf{x}^T S\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$ is λ_1.
   (Equivalently, this gives that the maximum value of the quadratic form $P(\mathbf{x}) = \mathbf{x}^T S\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ is λ_1.)

3. (Strang I.7.21) Draw the tilted ellipse $x^2 + xy + y^2 = 1$ and find the half-lengths of its axes from the eigenvalues of the corresponding symmetric matrix S.

4. (Strang I.9.2) Find a closest rank-1 approximation to the following matrices:

   $$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}, \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

   Following up on the third matrix: what are the singular values of a symmetric matrix, in general?

   On the fourth: what about of an orthogonal matrix?

5. (Strang I.12.10) Suppose all entries are 1 in a $2 \times 2 \times 2$ tensor T, except the first entry $t_{111} = 0$. Write T as a sum of two rank-1 tensors.

   Solve the following optimization problem:

   Find the closest rank-1 tensor $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \circ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \circ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to T (in the Frobenius norm), assuming that $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   (This is a least squares problem – can you see how to frame it that way?)

6. (Strang I.12.8) The largest possible rank of a $2 \times 2 \times 2$ tensor is 3. Can you find an example?