1. Prove the following facts about matrix ranks:
   i. \( \text{rank}(AB) \leq \text{rank}(A) \) and \( \text{rank}(AB) \leq \text{rank}(B) \)
   ii. \( \text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B) \)
   iii. \( \text{rank}(A^T A) = \text{rank}(AA^T) = \text{rank}(A) = \text{rank}(A^T) \) \( \text{(Hint: recall HW1 #3)} \)

2. (Strang I.3.4) If \( S = S^T \) is symmetric then of course \( C(S) = C(S^T) \) (i.e. the column and row spaces of \( S \) coincide) and also \( N(S) = N(S^T) \). Does the converse hold? That is, if \( A \) is square with \( C(A) = C(A^T) \) and \( N(A) = N(A^T) \), is \( A \) necessarily symmetric? Either prove the answer is yes or find a counterexample.

3. (Trefethen and Bau 2.1) Show that if a square matrix \( A \) is both triangular and orthogonal, then it is diagonal.

4. (Strang I.5.4) Suppose \( Q \) is \( n \times n \) and orthogonal. Check that \( ||Qx|| = ||x|| \) for any \( x \in \mathbb{R}^n \) (so orthogonal matrices don’t change lengths of vectors). Check that in fact \( (Qx)^T (Qy) = x^T y \) for any \( x, y \in \mathbb{R}^n \) (so orthogonal matrices don’t change angles between vectors).

5. (Trefethen and Bau 2.4) What are the possible eigenvalues of an orthogonal matrix \( Q \)? Remember, though its entries are real, \( Q \) may have complex eigenvalues.

6. Prove that eigenvectors associated to distinct eigenvalues of an orthogonal matrix \( Q \) are orthogonal. Remember, vectors \( x, y \) with possibly complex entries are orthogonal when \( x^* y = 0 \), where \( x^* = x^T \) is the complex conjugate of the transpose of \( x \).

7. (Trefethen and Bau 2.3) Suppose that \( S = S^T \) is a symmetric matrix (entries from \( \mathbb{R} \)).
   i. Prove that all eigenvalues of \( S \) are real. Conclude that for every eigenvalue we can find an associated eigenvector which is real.
      \( \text{(Hint: use the identity (AB)* = B*A* and the symmetry of S to prove \( \lambda x^* x = \lambda x^* x \) for a given eigenvalue \( \lambda \) and an associated eigenvector \( x \). For the conclusion about real eigenvectors, it may help to use that \( Sx = \lambda x \) iff \( (S - \lambda I)x = 0 \).) } \)
   ii. Prove that eigenvectors associated to distinct eigenvalues of \( S \) are orthogonal.
      \( \text{(One way: say why this holds when the first eigenvalue \( \lambda_1 = 0 \). In the general case consider the shifted matrix \( S - \lambda_1 I \).) } \)

8. (Strang I.6.22) Consider the following matrix:
   \[
   A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}
   \]
   Diagonalize \( A \) (i.e. factor \( A = XDX^{-1} \) where \( D \) is diagonal) and use this diagonalization to find a formula for \( A^k \).

9. (Strang I.6.12) The matrix \( A \) below is singular of rank one. Find three eigenvalues and three corresponding eigenvectors for \( A \):
   \[
   A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.
   \]