Homework #1

1. (Strang I.1.10-12) Find the CR factorizations of A_1 and A_2:

$$A_1 = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ 2 & 6 & -4 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

   What are the ranks of these matrices?

2. (Strang I.1.21) This problem is about the factorization $A = \mathbf{CMR}$, a close relative of $A = CR$. As before, $\mathbf{C}$ consists of the first r independent columns of A, where $r = \text{rank}(A)$. The new matrix $\mathbf{R}$ consists of *the first r independent rows of A*. The $r \times r$ "mixing matrix" $\mathbf{M}$ is the matrix that makes $A = \mathbf{CMR}$ a true equation.

   We can get an equation for $\mathbf{M}$ by observing: if $A = \mathbf{CMR}$, then $\mathbf{C^T A R^T} = \mathbf{C^T C M R R^T}$. The matrices $\mathbf{C^T C}$ and $\mathbf{RR^T}$ are always invertible (see Problem 3 below). Hence:

$$\mathbf{M} = (\mathbf{C^T C})^{-1}(\mathbf{C^T} A \mathbf{R^T})(\mathbf{RR^T})^{-1}.$$

   Use this equation to factor the following matrix A as $\mathbf{CMR}$:

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

   It may help to recall the formula for the inverse of a 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

3. (Strang I.3.6) Prove that $A^T A$ has the same nullspace as A.

   Hint: The containment $N(A) \subseteq N(A^T A)$ is easy. To prove $N(A^T A) \subseteq N(A)$, use the fact that $\mathbf{x}^T A^T A \mathbf{x} = ||A\mathbf{x}||^2$.

   This problem shows that if an $n \times m$ matrix A has full rank m, then the $m \times m$ symmetric matrix $A^T A$ is invertible!

4. (Trefethen and Bau 1.3) Suppose that U is an invertible $n \times n$ upper triangular matrix. Argue that U^{-1} is also upper triangular.

   Hint: Use the fact that a square matrix is invertible if and only if its columns are independent. As a first step, what can you say about the diagonal entries in U?

5. (Trefethen and Bau 1.4) Let $f_1, \ldots, f_8$ be a set of functions defined on the closed interval $[1, 8]$ with the property that for any real numbers $d_1, \ldots, d_8$, there exists a set of coefficients $c_1, \ldots, c_8$ such that

$$\sum_{j=1}^{8} c_j f_j(i) = d_i, \quad i = 1, \ldots, 8.$$

   a. Express the above as a matrix equation $A\mathbf{c} = \mathbf{d}$, and write down A.

   b. Say why the numbers $d_1, \ldots, d_8$ determine the coefficients $c_1, \ldots, c_8$ *uniquely*.

6. (Strang I.4.6) Consider the following matrix A:

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

a. Find a c for which the factorization $A = LU$ is possible, and factor.

b. Which c leads to a 0 in the second pivot position? Then a row exchange is needed corresponding to a permutation matrix P, and $A = LU$ is not possible. In this case, factor PA as LU.

c. Which c produces 0 in the third pivot position? Then a row exchange cannot help and elimination fails.