Some of HW8

<u>Notation</u>: $A \cong B$ means A is isomorphic to B.

3.19a. Suppose our language contains a single unary relation symbol S.

Prove there is a countable family $\mathcal{F}$ of countable structures, such that *every* countable structure in this language is isomorphic to a structure in the family.

Prove also that the structures in the family $\mathcal{F}$ are pairwise non-isomorphic.

Proof: For $n \in \omega$, define $A(n, \infty)$ to be the structure $(|A(n,\infty)|, S^{A(n,\infty)}) = (\omega, \{0, 1, \ldots, n-1\})$

For $m \in \omega$ define $A(\infty, m)$ to be the structure $(|A(\infty, m)|, S^{A(\infty,m)}) = (\omega, \{m, m+1, \ldots\})$.

Define $A(\infty, \infty) = (\omega, \{0, 2, 4, \ldots\})$.

<u>Claim 1</u>: If A is a countably infinite structure in this language, then there are $x, y \in \{0, 1, \ldots, \infty\}$ such that A is isomorphic to $A(x, y)$.

Proof: There are three possibilities: S^A is finite, $|A| - S^A$ is finite, both A and S^A are infinite.

Suppose we are in case 1, and S^A is of size n. Choose an enumeration $|A| = \{a_0, a_1, \ldots, a_{n-1}, a_n, \ldots\}$ so that $S^A = \{a_0, \ldots, a_{n-1}\}$ Define $\pi : A(n, \infty) \to A$ by $\pi(i) = a_i$.

Then π is an isomorphism since it is a bijection and $i \in S^{A(n,\infty)}$ iff $i \in \{0, \ldots, n-1\}$ iff $a_i \in \{a_0, \ldots, a_{n-1}\}$ iff $\pi(i) \in \{\pi(a_0), \ldots, \pi(a_{n-1})\}$ iff $\pi(i) \in S^A$.

Similarly for the other cases.

<u>Claim 2</u>: The structures in $\mathcal{F}$ are pairwise non-isomorphic.

Proof: Fix $A(x, y)$ and $A(x', y')$ in our family s.t. $(x, y) \neq (x', y')$. (at least one of x, y is ∞ and at least one of x', y' is ∞.)

WLOG $x \neq x'$ and $x < x'$. Hence x is finite, say $x = n$. Then $S^{A(x,y)} = \{0, \ldots, n-1\}$.

Let $\pi : |A(x', y')| \to |A(x, y)|$ be any bijection. We know $S^{A(x,y)} = \{0, \ldots, n-1\}$ And $S^{A(x',y')} = \{0, \ldots, n-1, n, \ldots\}$ is of size $x' > n$ (x' possibly infinite).

Hence $\{\pi(0), \ldots, \pi(n-1), \pi(n), \ldots\}$ is of size x' as well. Thus there must be some N such that $N \in S^{A(x',y')}$ but $\pi(N) \notin S^{A(x,y)}$. Hence π is not an isomorphism. Since π was arbitrary, there is no isomorphism.

3.19b Consider the language with a single binary relation symbol R. Construct a family of uncountably many pairwise non-isomorphic countable structures in this language.

Proof: First, an example.

A useful way to think about isomorphisms is: if A, B structures and $\pi : |A| \to |B|$ a bijection then π is an isomorphism if when you "apply π" to c^A, R^A, f^A for all the symbols in your language you get c^B, R^B, f^B.

Consider the structures in this language $A = (|A|, R^A) = (\{1, 2, 3\}, (1, 2), (1, 3))$, $B = (|B|, R^B) = (\{1, 2, 3\}, \{(2, 3), (2, 1)\})$, $C = (|C|, R^C) = (\{1, 2, 3\}, \{(1, 1), (2, 2)\})$.

Then A is isomorphic to B. Bijection is given by $\pi(1) = 2$, $\pi(2) = 3$, $\pi(3) = 1$; is an isomorphism because when you apply π to $R^A = \{(1, 2), (2, 3)\}$ you get $\{(2, 3), (3, 1)\} = R^B$.

But A is not isomorphic to C. For any bijection π from $\{1, 2, 3\}$ we have

"$\pi[R^A]$" $= \{(\pi(1), \pi(2)), (\pi(1), \pi(3))\} \neq R^C$.

Now we prove the problem.

For every infinite $X \subseteq \omega$, we list X in increasing order: $X = \{n_0, n_1, \ldots\}$

There are uncountably many infinite subsets of ω.

For every such X, we define a relation

$R^{A_X} = \{(0,0), (0,1), \ldots, (0, n_0 - 1), (1,0), (1,1), \ldots, (1, n_1 - 1), \ldots\}$

The point: for every $k \in \omega$, there are exactly n_k many tuples of the form $(k, \cdot)$ in the relation.

Notice: if $k < l$ then than number of tuples $(k, \cdot)$ is n_k which is less than n_l which is the number of tuples of the form $(l, \cdot)$

E.g. if $X = \{2, 4, 6, \ldots\}$ Then $R^{A_X} = \{(0,0), (0,1), (1,0), (1,1), (1,2), (1,3), ...\}$

We now define a structure A_X with $|A_X| = \omega$ and R^{A_X} as just defined.

Claim: if $X \neq Y$ then A_X is not isomorphic to A_Y.

Proof: We write $X = \{n_0, n_1, \ldots\}$, $Y = \{m_0, m_1, \ldots\}$ in increasing order.

Wlog there is $n \in A_X$ such that $n \notin A_Y$. Then $n = n_k$ for some k. Hence the number of tuples of the form $(k, \cdot)$ in R^{A_X} is $n_k = n$.

If there were an isomorphism $\pi : A \to B$ we would have to have that number of tuples of the form $(\pi(k), \cdot)$ in R^{A_Y} is n also.

But since $n \notin Y$, for every k we have that the number of tuples of the form $(k, \cdot)$ in R^{A_Y} is $m_k \neq n$

Hence there is no isomorphism, i.e. A_X and A_Y not isomorphic.

(i)

Interlude: More on structures + isomorphism.

<u>Graphs</u>

- Consider lang. w/ single binary relation symbol R.
- a <u>graph</u> is a structure $\mathcal{A}$ satisfying the following theory Σ:

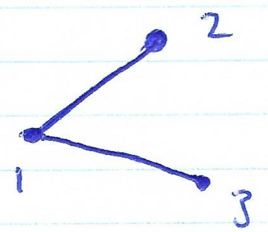
$$\Sigma = \{ \forall u \, \neg R(u,u), \; \forall u \forall v \, (R(u,v) \Rightarrow R(v,u)) \}$$

- we say: a graph is a set equipped w/ an irreflexive, symmetric relation

- e.g. $\mathcal{A} = (|\mathcal{A}|, R^{\mathcal{A}})$
$$= (\{1,2,3\}, \{(1,2), (2,1), (1,3), (3,1)\})$$

is a graph

<u>Pic</u>:



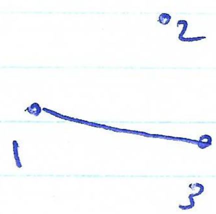
draw an edge between a, b iff $(a,b) \in R$

$\mathcal{A}$

We say x, y are <u>adjacent</u> in a graph $\mathcal{A}$ iff $(x,y) \in R^{\mathcal{A}}$.

(ii)

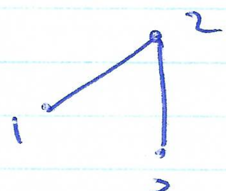
Another graph:



A'

$R^{A'} = \{(1,3),(3,1)\}$

Another:



A''

$R^{A''} = \{(1,2),(2,1)$
$(2,3),(3,2)\}$

Observe: $A \cong A''$ but $A \not\cong A'$ (why?)

Consider $B = (|B|, R^B)$
$= (\{1,2,3\}, \{(1,2),(2,1),(1,3)\})$

Then B is a structure in this lang. but is not a graph!
(R^B not symmetric)

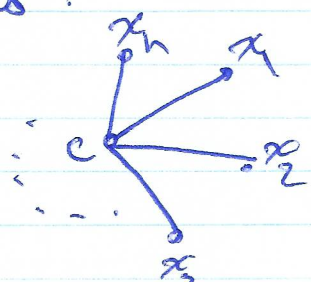
OTOH: $C = (|C|, R^C)$
$= (\{1,2\}, \{(1,2),(2,1)\})$
is a graph
– in fact C is a substructure of A, since $|C| \subseteq |A|$
$R^C = R^A \upharpoonright |C|$

(iii)

Note: A' above is not a substructure of A since $|A'| = |A|$ but $R^{A'} \neq R^{A}$.

Claim: There is an unctbl family $\mathcal{F}$ of pairwise non-isomorphic ctbly infinite graphs

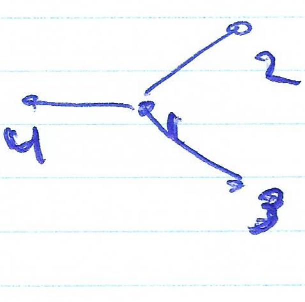
Pf. For $n \in \omega$, $n \geq 1$ an n-star is a graph that looks like this:



the point:
in an n-star:
center is adjacent to n points;
all other pts adjacent only to center.

e.g. $A = (|A|, R^{A})$
$= (\{1,2,3,4\}, \{(1,2),(2,1),$
$(1,3),(3,1),$
$(1,4),(4,1)\})$

is a 3-star:

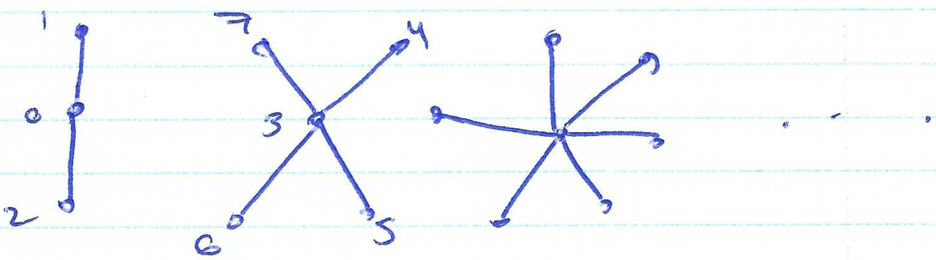


(iv)

Sps $X = \{n_0, n_1, \ldots\} \subseteq \omega$ is an infinite subset of ω not containing 0.

Let A_X be the graph consisting of infinitely many stars, one for each $n \in X$.

e.g. if $X = \{2, 4, 6, \ldots\}$, A_X looks like



Actually defining A_X explicitly isn't so important, could do:

$$A_X = (\omega, \{(0,1),(1,0),(0,2),(2,0),(3,4),(4,3),(3,5),(5,3),(3,6),(6,3),(3,7),(7,3) \ldots\})$$

Claim: if $X \neq Y$ then A_X and A_Y are not isomorphic.

Pf. wlog there is $n \in X$ s.t. $n \notin Y$

(v)

Hence there is $x \in |A|_x$ that is adjacent to exactly n-many points (center of n-star)

For any $y \in |A_y|$, adjacent to m points for some $m \in Y$, or adjacent to exactly 1.

Hence $A_x \not\cong A_y$ ✓

Hence $\mathcal{F} = \{A_x : x \subseteq \omega\setminus\{0\} \text{ infinite}\}$ is as desired.

Linear orders

A linear order is any structure $\mathcal{A}$ satisfying the theory:

$$\forall u\,(\neg R(u,u))$$
$$\forall u \forall v\,(R(u,v) \Rightarrow \neg R(v,u))$$
$$\forall u \forall v \forall w\,(R(u,v) \wedge R(v,w) \Rightarrow R(u,w))$$

(L.O.'s are sets w/ irreflexive antisymmetric transitive relations)

This def'n is for strict orders

e.g. $(\mathbb{R}, <)$ is a linear order by this def'n but $(\mathbb{R}, \leq)$ is not.

(vi)

- Visualize an L.O. by drawing points in a line: if $(a,b) \in R$ then a left of b. No edges!

e.g. $\mathcal{A} = (A, R^{\mathcal{A}})$
$= (\{1,2,3\}, \{(1,2),$
$(2,3)$
$(1,3)\})$

is a l.o.

$\mathcal{A}$: $\quad \bullet \quad \bullet \quad \bullet$
$\quad\quad 1 \quad 2 \quad 3$

- Now consider $\mathcal{A} = (\omega, <)$

$\mathcal{A}$: $\quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \dots$
$\quad\quad 0 \quad 1 \quad 2 \quad 3$

- And $\mathcal{B} = (\mathbb{Z}, <)$

$\dots \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \dots$
$\quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

Then $|\mathcal{A}| \subseteq |\mathcal{B}|$ and $R^{\mathcal{A}} = R^{\mathcal{B}} \upharpoonright |\mathcal{A}|$
so $\mathcal{A}$ is a substructure of $\mathcal{B}$

(vii)

- Is A an elementary substructure of B?

  No: $A \models \exists u \forall v (\neg R(v,u))$

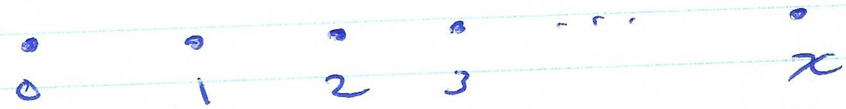
  $B \not\models$ " "

- Consider $C = (|C|, R^C)$

  where $|C| = \omega \cup \{x\}$

  $R^C = R^A \cup \{(n,x) : x \in \omega\}$

Picture:



- Then A is a substructure in C
- Is A elementary in C?

  No again:

  $A \models \forall u \exists v (R(u,v))$

  elements have successors

  $C \not\models$ " "

  since x has no successor.

(viii)

What about C':

$$\begin{array}{ccccccccc} \circ & \circ & \circ & \cdots & \circ & \circ & \circ & \cdots \\ 0 & 1 & 2 & & x_0 & x_1 & & \end{array}$$

Is A elementary in C?

<u>Still no</u>:

$$A \models \exists u \forall v \left(u \neq v \Rightarrow \left[R(u,v) \wedge \exists w \left(R(w,v) \wedge \forall x \neg \left(R(w,x) \wedge R(x,w) \right) \right) \right] \right)$$

"A has a unique minimal element. Every other element has a unique predecessor"

$C' \not\models$ " "

x_0 has <u>no</u> predecessor

However we will prove (later):

<u>Thm</u>: there is a linear order B s.t.

- $A = (\omega, <)$ is an elem substructure of B
- $\exists x \in |B|$ s.t. for all $n \in \omega$ $n < x$.

(ix)

- For now something weaker
- For every $n \in \omega$ let c_n be a new constant
- Let A^* be expansion of $A = (\omega, <)$ that interprets $c_n^{A^*} = n$
- Let T^* be the set of <u>all</u> sentences φ s.t. $A^* \models \varphi$.

e.g. T^* contains axioms for a linear order, indeed <u>all</u> sentences true in A, as well as the following:

- $\forall u \, (u \not< c_0)$
- Sentences of the form

$$c_n < c_{n+1} \wedge \neg \exists u \, (c_n < u \wedge u < c_{n+1})$$

Hence any model $B \models T^*$ is a linear order that looks like ω "at the beginning"

$$c_0^B \quad c_1^B \quad c_2^B \quad \ldots$$

<u>Question</u>: if $B \models T^*$ is B isomorphic to A^*? $= (\omega, <, \ldots)$

(x)

- We prove no!

- Let c be a new constant symbol
Let Σ be the set of sentences
of the form $c_n < c$

- We prove $T^* \cup \Sigma$ is satisfiable.

- Let $\Delta \subseteq T^* \cup \Sigma$ be finite
$= \Delta_0 \cup \Delta_1$
where $\Delta_0 \subseteq T^*$
$\Delta_1 \subseteq \Sigma$

- Let $c_{n_0}, \ldots, c_{n_k}$ be set of c_n's
appearing in Δ_1

- Let $N = n_k + 1$

- Let A' be the expansion of
A^* that interprets c as N

- Then $A' \models \Delta_0$ because actually
$A' \models T^*$

- and $A' \models \Delta_1$ since $c^{A'} = N$ really
is larger than $c_n^{A'} = n$ for all
c_n's appearing in Δ_1

- Hence by compactness $T^* \cup \Sigma$
is satisfiable

(xi)

- Let B be a model. Then:

- Since $B \models T^*$ we know every element in $|B|$ except c_0^B has a unique successor and predecessor

So