(i) - elem substruct
- notation $\preceq$ $\preceq$

Upward Löwenheim - Skolem.

- Downward LS says: in a big (i.e. unctbl) structure B can always find a small (i.e. ctbl) ~~structure~~ elem. substructure A

- Upward LS intuitively says: given a ("small") structure A, can always find a structure B of larger size s.t. A is elementary in B.

- need some preliminaries before we can prove.

- Let A be a fixed structure
- For every $x \in |A|$ introduce a new constant c_x.
- Let D be the expansion of A that interprets each c_x as x. We write:

$$D = (A, x)_{x \in |A|}$$

- The elementary diagram of A is the theory

$$\mathrm{Diag}(A) = \{\varphi \mid D \models \varphi\}$$

- Observe: $\mathrm{Diag}(A)$ is complete and actually for every sentence φ in expanded lang exactly one of $\varphi, \neg\varphi$ is in $\mathrm{Diag}(A)$

(ii)

- And clearly $D \models Diag(A)$.

Lemma 4.6 - Sps A is a structure and $D = (A, x)_{x \in |A|}$.

- Sps $E \models diag(A)$
- Let B be reduct of E to original language of A
- Then the map:

$$\pi : A \to B$$

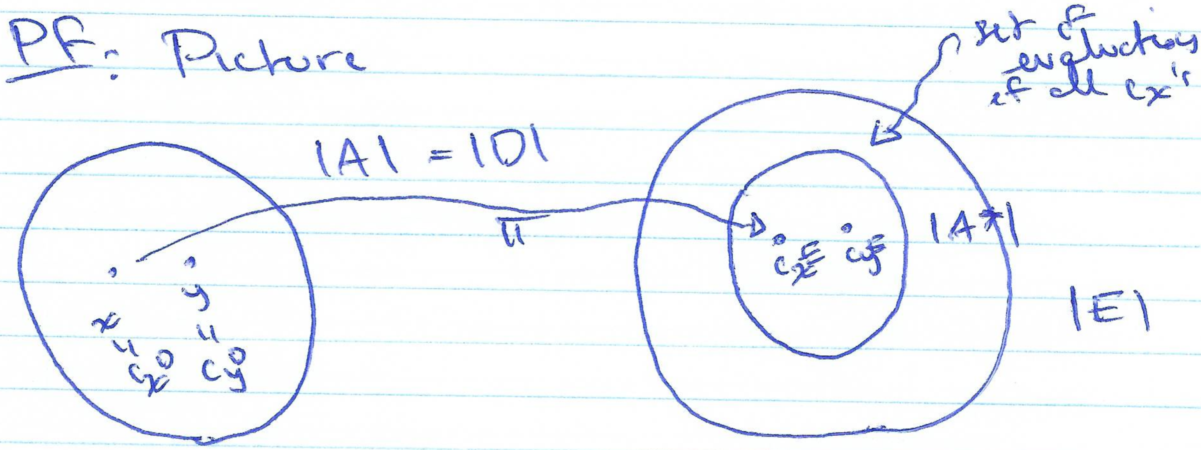
defined by

$$\pi(x) = c_x^E \quad (= c_x^B)$$

is an isomorphism of A onto an elementary substructure A^* of B

↳ intuitively says any model of Diag(A) contains an (elementary) copy of A

PF: Picture



(iii)

Claim 1: π is injective (hence a bijection of $|A|=|D|$ with $\mathrm{ran}(\pi)$)

Pf: Sps $x, y \in |D|^{=|A|}$ and $x \neq y$

Then $D \models c_x \not\approx c_y$ i.e. $c_x \not\approx c_y \in \mathrm{Diag}|A|$

Hence $E \models c_x \not\approx c_y$

i.e. $c_x^E \neq c_y^E$

i.e. $\pi(x) \neq \pi(y)$ ✓

– next want to show that $\mathrm{ran}(\pi)$ is universe of a substructure of E

Claim 2: For every constant d in expanded lang. we have $\pi(d^D) = d^E$

Pf: There is $x \in |A|^{=|D|}$ s.t. $d^D = x$

Hence $c_x^D = x$

Hence $D \models c_x \approx d$

Hence $E \models c_x \approx d$

i.e. $d^E = c_x^E$

i.e. $d^E = \pi(x) = \pi(d^D)$ ✓

Claim 3: For a function symbol F in lang and $x_1, \ldots, x_n \in |A| = |D|$ we have

$$F^D(x_1, \ldots, x_n) = y$$

iff $F^E(\pi(x_1), \ldots, \pi(x_n)) = \pi(y)$

Pf: $F^D(x_1, \ldots, x_n) = y$

iff $F^D(c_{x_1}^D, \ldots, c_{x_n}^D) = c_y^D$

iff $F(c_{x_1}, \ldots, c_{x_n}) = c_y \in \mathrm{Diag}(A)$

(iv)

iff $F^E(c_{x_1}^E, \ldots, c_{x_n}^E) = c_y^E$
iff $F^E(\pi(x_1), \ldots, \pi(x_n)) = \pi(y)$

Claim 4: For R an n-ary relation symbol and $x_1, \ldots, x_n \in |A| = |D|$ we have
$$R^D(x_1, \ldots, x_n) \text{ iff } R^E(\pi(x_1), \ldots, \pi(x_n))$$

pf: similar to 2,3.

Observe: claims 2,3,4 show $\mathrm{ran}(\pi)$ is closed under constants and functions, hence we may define the substructure D^* of E whose universe is $\mathrm{ran}(\pi)$.

2,3,4 also show that π is an isomorphism of D with D^*.

Now: - Let A^* be the reduct of D^* to original lang w/o new constants.
- Let B be reduct of E to original lang.
- Then A^* is a substructure of B and π is an isomorphism of A with A^*.
- we want to show A is elementary in B.

(v)

- We first show D^* is elem in E
- By T-V enough to show for every fmla $\varphi(v)$ in expanded lang:

  if $E \models \exists v\, \varphi(v)$

  then there is $y \in |D^*| = \mathrm{ran}(\pi)$

  s.t. $E \models \varphi(y)$

- So fix $\varphi(v)$ and sps $E \models \exists v \varphi(v)$
- then $\exists v \varphi(v) \in \mathrm{Diag}(A)$
- hence $D \models \exists v \varphi(v)$
- i.e. there is $x \in |D| = |A|$ s.t.

  $D \models \varphi(x)$

or really $D \models \varphi(c_x)$ ← sentence

hence $\varphi(c_x) \in \mathrm{Diag}(A)$

hence $E \models \varphi(c_x)$

but then $y = c_x^E$ works since $c_x^E \in |D^*|$

↳ Hence $D^* \preceq E$ as claimed

↳ it follows easily that A^* is elem in B:

- if $\varphi(v)$ is fmla in original lang then if

  $B \models \exists v \varphi(v)$, then we have

  $E \models \exists v \varphi(v)$, which gives

  $\exists v \varphi(v)$ is in $\mathrm{Diag}(A)$

hence there is some $x \in |A| = |D|$

s.t. $D \models \varphi(c_x)$

hence $D^* \models \varphi(c_x)$

hence $A^* \models \varphi(c_x^E)$ ← in $|A^*|$

(vi)

Ex 1: if A is finite structure (i.e. $|A|$ is finite) and $E \models \mathrm{Diag}(A)$ and B is reduct of E to original lang then A is actually isomorphic to B

Pf. if $|A| = \{x_1, \dots, x_n\}$ then $\mathrm{Diag}(A)$ contains sentence
$\forall v\,(v \approx c_{x_1} \vee v = c_{x_2} \vee \dots \vee v = c_{x_n})$

Hence $|E| = |B| = \{c_{x_1}^E, \dots, c_{x_n}^E\}$

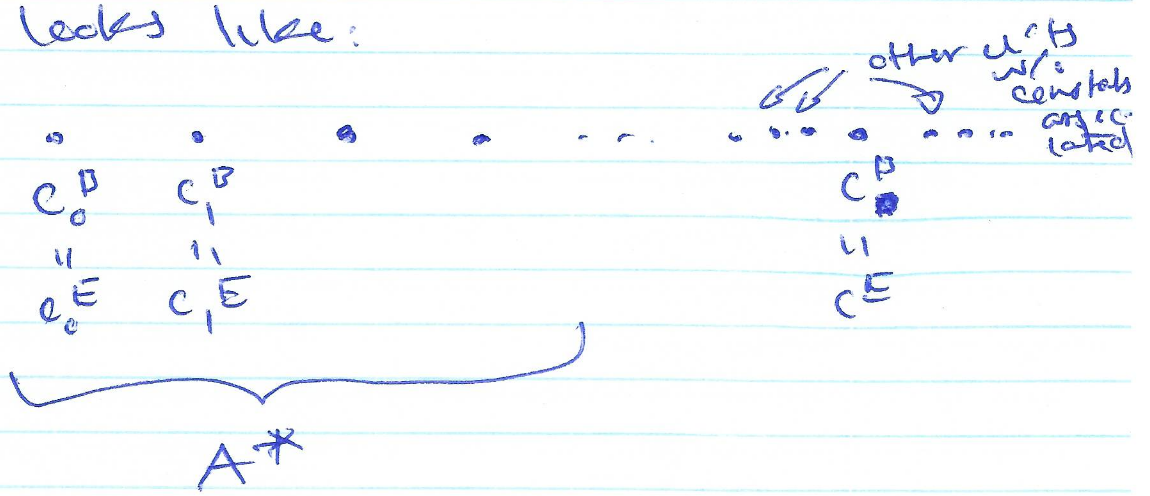
and the π defined in Lemma 4.6 is actually an isomorphism (of $D = (A, x)_{x \in |A|}$ with E and also of A with B)

Ex 2 Last time:
- if $A = (|A|, R^A) = (\omega, <)$ then there is a structure E s.t. $E \models \mathrm{Diag}(A) \cup \{c_n < c \mid n \in \omega\}$ where c is a new constant symbol.
- Let B be reduct of E to original lang (w/ only symbol R)
- Let A^* be substructure of this model w/ universe $\{c_n^E : n \in \omega\}$

- Then 4.6 tells us A^* is elementary substructure of B

that is isomorphic to $A = (\omega, <)$

B looks like:



- $A = (\omega, <) = \overset{\bullet}{0}\ \ \overset{\bullet}{1}\ \ \overset{\bullet}{2}\ \cdots$

is not literally an elem substructure of B, but is isomorphic to an elem substructure of B (namely A^*)

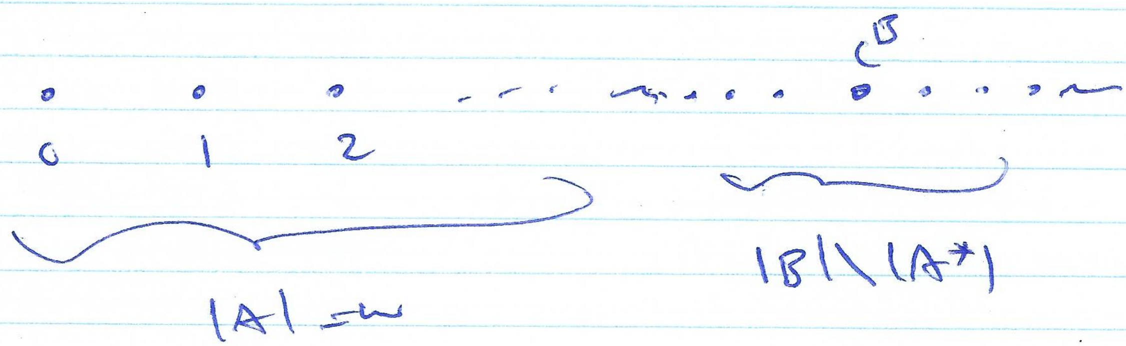
- We can define a new structure which "looks just like" B but in which A is literally an elementary substructure.

- Let B' be structure with universe $(|B| \setminus |A^*|) \cup |A|$ "ω"

(viii)

- Define $R^{\mathcal{B}'}$ as follows:
  - if $n, m \in |\mathcal{A}| = \omega$ then $R^{\mathcal{B}'}(n,m)$ iff $R^{\mathcal{A}}(n,m)$
  - if $x, y \in |\mathcal{B}| \setminus |\mathcal{A}^*|$ $R^{\mathcal{B}'}(x,y)$ iff $R^{\mathcal{B}}(x,y)$
  - if $n \in |\mathcal{A}|$, $x \in |\mathcal{B}| \setminus |\mathcal{A}^*|$
    $R^{\mathcal{B}'}(n,x)$ iff $R^{\mathcal{B}}(c_n^{\mathcal{B}}, x)$
    $R^{\mathcal{B}'}(x,n)$ iff $R^{\mathcal{B}}(x, c_n^{\mathcal{B}})$

$\mathcal{B}'$:



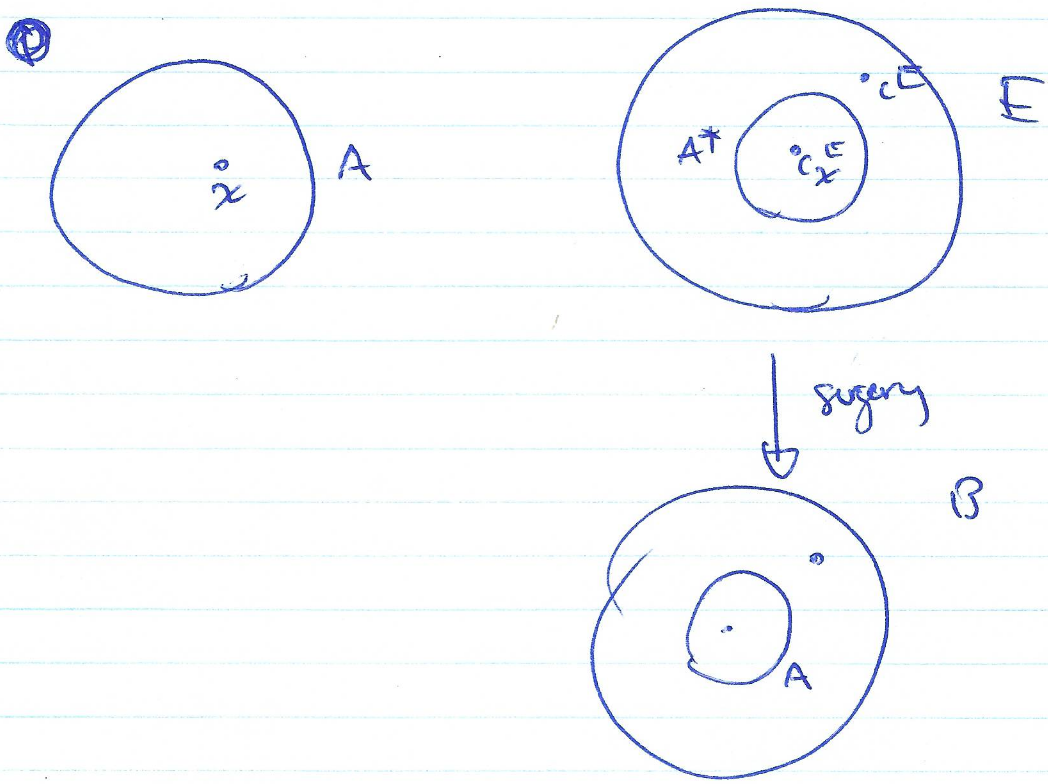
Then $\mathcal{B}'$ is isomorphic to $\mathcal{B}$ and $\mathcal{A}$ is literally an elementary substructure of $\mathcal{B}'$ (we call $\mathcal{B}$ an elementary extension of $\mathcal{A}$)

Can do this in general:

Lemma 4.7: If A is an infinite structure then A has a proper elementary extension B (i.e. $A \preceq B$ and $\exists x \in |B|$ $x \notin |A|$)

Strategy: - use compactness to build E that contains a copy A^* of A as elem substructure, and some $c^E \notin A^*$

- replace A^* with A "surgically" as in example above



(x)

PF: - Let $D = (A, x)_{x \in |A|}$
- Let c be a new constant symbol diff. from all the c_x's
- Consider theory:

$$\Sigma = \text{Diagram}(A) \cup \{c \neq c_x : x \in |A|\}$$

Claim Σ is finitely satisfiable

PF: - Fix $\Delta \subseteq \Sigma$ finite

$\Delta = \Delta_0 \cup \Delta_1$ where $\Delta_0 \subseteq \text{Diag}(A)$, $\Delta_1 \subseteq \{c \neq c_x\}$

- There is $x_0 \in |A|$ s.t. $c \neq c_{x_0} \notin \Delta_1$ (because $|A|$ is infinite!)

- Let D_Δ be expansion of D that interprets c as x_0

- Then $D_\Delta \models \Delta_0$ since actually $D \models \Sigma$ and $D \models \Delta_1$ since $c^D = x_0$ really is diff from c_x^D for all c_x's appearing in Δ_1

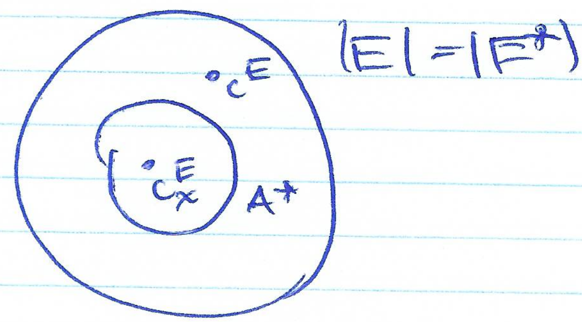
- Hence $D_\Delta \models \Delta$
- Since Δ was arbitrary, Σ is finitely satisfiable ✓

- Hence Σ has a model, E, by compactness

- Let E^* be reduct of E to lang of A

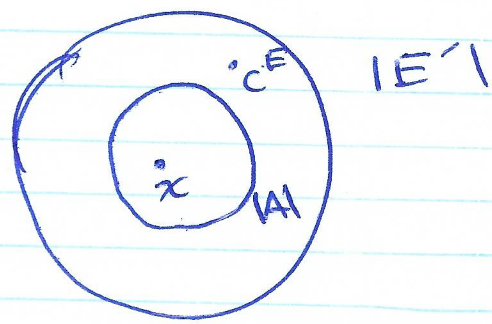
(xi)

- Since $E \models Diag(A)$ we know $\{c_x^E : x \in |A|\}$ is universe of an elem. substructure of E^* iso to A
- call this substructure A^*
- we know $|E| = |E^*|$ contains an element $c^E \notin |A^*|$



- We don't literally have $A \leq E^*$ only $A^* \leq E^*$.
- We define a new struct E' by "replacing A^* by A"
- let $X = |E^*| \setminus |A^*| \cup |A|$
- let E' be struct in lang of A with $|E'| = X$ where every symbol interpreted ~~[illegible]~~ as you'd guess
- i.e. if d is a constant symbol and $d^E = c_x^E$ (it has to be c_x^E for some x since $d^A = x$) then let $d^{E'} = x$

For symbols R, F to define $R^{E'}$ and $F^{E'}$ look at R^E and F^E and everywhere you see a c_x^E replace with x.



- This defines a structure E'
- By construction A is literally a substructure
- and by construction we have $A \preceq E'$
- hence $B = E'$ $\blacksquare$ works ✓

↳ So unlike finite structures infinite structures A can't be
• determined up to isomorphism by their diagram

↳ always a bigger structure B s.t. $A \preceq B$.

(xiii) 37

↪ nothing special about adding one new constant symbol

<u>Upward</u> Löwenheim - Skolem Theorem

Let A be a structure. Then A has an unctbl elementary extension.

<u>Pf</u>: - Let $D = (A, x)_{x \in |A|}$
- Let I be an unctbl indexing set (e.g. could take $I = P(\omega)$)
- For each $i \in I$ introduce a new constant symbol c_i.
- Consider the theory Σ:

$$\Sigma = \mathrm{Diag}(A) \cup \{c_x \not\approx c_i : x \in |A|, i \in I\} \cup \{c_i \not\approx c_j : i \neq j\}$$

- Then Σ is finitely satisfiable (why?)
- Hence by compactness Σ has a model E
- Let E^* be reduct of E to lang. of A
- Then E^* contains an elementary substructure A^* isomorphic to A

(xiv)

- let $\mathfrak{E}'$ be structure obtained by "replacing" A^* by A
- then $\mathfrak{E}'$ is uncbl and $A \subseteq \mathfrak{E}'$ ✓

↳ Can view LS theorems as theorems about "limited expressive power" of FOL wrt infinite sets

↳ FOL is strong enough to do a lot of nice mathematics in but not ~~[illegible]~~ allowed to quantify over infinite subsets of universe in FOL.

↳ in SOL you can, but lose all the powerful theorems of FOL: compactness, LS theorems etc.

(i)

Dense linear orders

- Consider the theory consisting of the following axioms in lang w/ a single binary relation symbol R:

axioms for linear orders:

$\forall u \; \neg R(u,u)$

$\forall u \forall v \forall w \; (R(u,v) \wedge R(v,w) \Rightarrow R(u,w))$

$\forall u \forall v \; (R(u,v) \vee R(v,u) \vee u = v)$

density: $\forall u \forall v \; (R(u,v) \Rightarrow \exists w (R(u,w) \wedge R(w,v)))$

no bottom pt: $\forall u \exists v \; R(v,u)$

no top pt: $\forall u \exists v \; R(u,v)$

- Any model $A = (|A|, R^A)$ of this theory will be a linear order that is dense and w/o endpoints
- This theory is denoted DLO.

<u>ex</u>: Consider the linear orders:

$A = (A, R^A) = (\mathbb{Q}, <)$

$B = (B, R^B) = (\mathbb{R}, <)$

$C = (C, R^C) = ((0,\infty), <)$
$= (\mathbb{R}^+, <)$

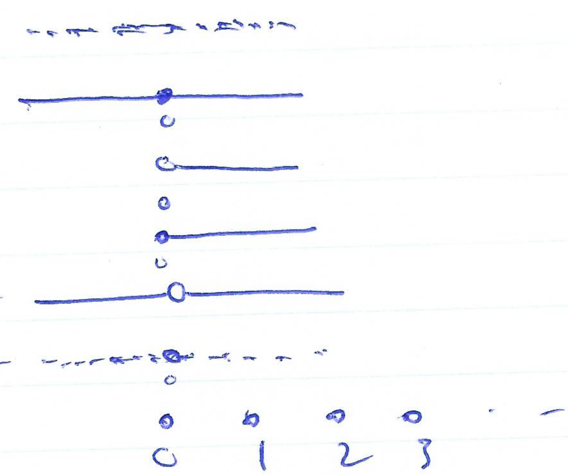
(ii)

$D = (D, R^D) = ([0,\infty), <)$
$E = (E, R^E) = (\mathbb{R}\setminus\{0\}, <)$
$F = (F, R^F) = (\mathbb{Q}\setminus\{0\}, <)$
$G = (G, R^G) = (\omega, <)$

- Then A, B, C, E, F are models of DLO
- D is not a model of DLO since it has a left endpoint
- G is not a model of DLO since it has a left endpoint <u>and</u> is not dense.

<u>Picture</u>:

$A = (\mathbb{Q}, <) =$
$B = (\mathbb{R}, <) =$
$C = (\mathbb{R}^+, <) =$
$D = ([0,\infty), <) =$
$E = (\mathbb{R}\setminus\{0\}, <) =$
$F = (\mathbb{Q}\setminus\{0\}, <) =$
$G = (\omega, <) =$



(iii)

- Which of these orders are isomorphic?
- A not iso to B since $|A| = \mathbb{Q}$ is ctbl and $|B| = \mathbb{R}$ is <u>not</u>
- B <u>is</u> isomorphic to C

<u>pf</u>: The map
$f: \mathbb{R} \to \mathbb{R}^+$
$f(x) = e^x$ is an order-preserving bijection of $\mathbb{R}$ with $\mathbb{R}^+$

- B is <u>not</u> isomorphic to D since D has a left endpoint and B does not (so they do not even satisfy same first order theory)

- is B isomorphic to E?

No: this is prob 4.2 from Ch. 4 exercises.

↳ it turns out however, B and E satisfy <u>the same</u> first-order sentences in the language w/ only R!

(iv)

Key fact: $(\mathbb{R}, <)$ has <u>least</u> <u>upper</u> <u>bound</u> <u>property</u>, which we now define.

- if A is a linear order and $S \subseteq |A|$ and $x \in |A|$ we say x is an <u>upper</u> <u>bound</u> for S iff $\forall a \in S$ we have $a \leq x$

- x is a <u>least</u> <u>upper</u> <u>bound</u> (l.u.b.) for S iff x is an upper bound and for any other upper bound y for S we have $x \leq y$.

- <u>Fact</u>: if $S \subseteq \mathbb{R}$ and S has an upper bound then S has a l.u.b. (can take for granted)

- e.g. if $S = \{x \in \mathbb{R} : x^2 \leq 2\}$ then $x = 12$ is an upper bound and $x' = \sqrt{2}$ is a lub.

- OTOH $T = \{q \in \mathbb{Q} : q^2 \leq 2\}$ has no l.u.b. in $(\mathbb{Q}, <)$.

  ↳ $\mathbb{Q}$ does not have l.u.b. property: another reason $(\mathbb{Q}, <)$ $(\mathbb{R}, <)$ not isomorphic

- what about $A = (\mathbb{Q}, <)$ and $F = (\mathbb{Q}\setminus\{0\}, <)$?

↳ turns out these orders are isomorphic

↳ this is an instance of a much more general fact.

Theorem (Cantor) Suppose $A = (|A|, R^A)$ and $B = (|B|, R^B)$ are models of DLO. Then A is isomorphic to B.

Pf. We build an isomorphism between A and B by going "back and forth"

enumerate: $|A| = a_0, a_1, \ldots$
$|B| = b_0, b_1, \ldots$

- We will define an iso $f: A \to B$ as the union of an increasing chain of partial isomorphisms $f_0 \subseteq f_1 \subseteq \ldots$

↳ $f = \bigcup f_n$

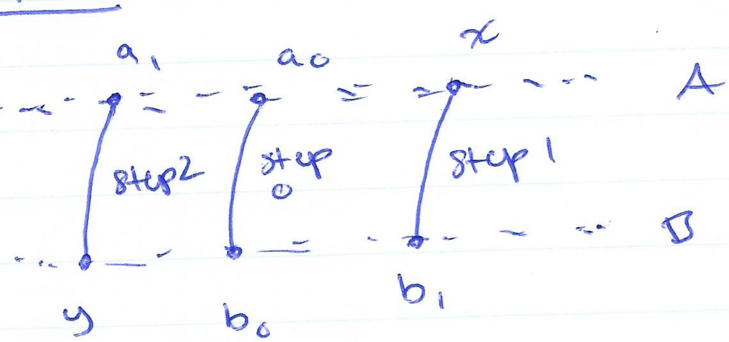
- need to arrange f is order-preserving and

(*) every $a \in |A|$ is in dom(f)

(**) every $b \in |B|$ is in ran(f)

- We ensure (*) by arranging at an even stage $2i$, we have $a_i \in \mathrm{dom}(f_{2i})$
- We ensure (**) by " " odd stage $2i+1$, we have $b_i \in \mathrm{ran}(f_{2i+1})$

Picture:



Step 0: Let $f_0 = \{(a_0, b_0)\}$

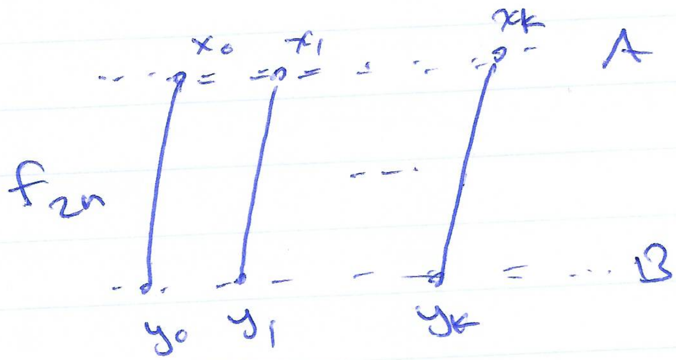
Step 2n+1: - Sps we have defined f_{2n} a partial order-preserving injection from $|A|$ to $|B|$

- list the elements of $\mathrm{dom}(f_{2n})$ in increasing order:

$$x_0 < x_1 < \dots < x_k$$

- list the el'ts of $\mathrm{ran}(f_{2n})$ in increasing order:

$$y_0 < y_1 < \dots < y_k$$



hence $f_{2n} = \{(x_0, y_0), \ldots, (x_k, y_k)\}$

<u>Now</u>:- if $b_n \in ran(f_{2n})$ just let $f_{2n+1} = f_{2n}$

- if $b_n \notin ran(f_{2n})$ there are three cases!

① $b_n < y_0$

② there is $i < k$ s.t. $y_i < b_n < y_{i+1}$

③ $y_k < b_n$

possible since no bottom pt

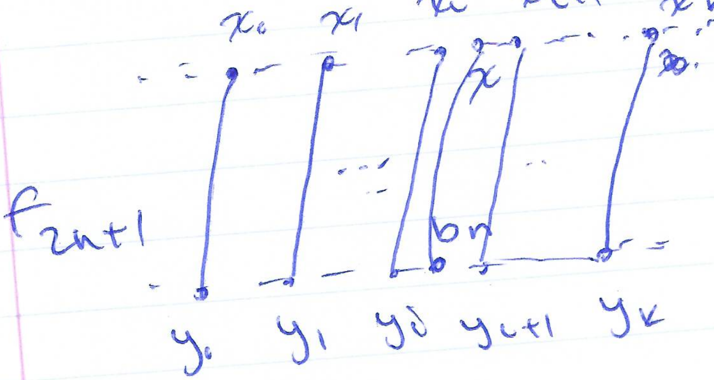
if ①: pick $x < x_0$ and let $f_{2n+1} = f_{2n} \cup \{(x, b_n)\}$

if ② pick x s.t. $x_i < x < x_{i+1}$ and let $f_{2n+1} = f_{2n} \cup \{(x, b_n)\}$

if ③ pick $x > x_k$ and let $f_{2n+1} = f_{2n} \cup \{(x, b_n)\}$

possible by density

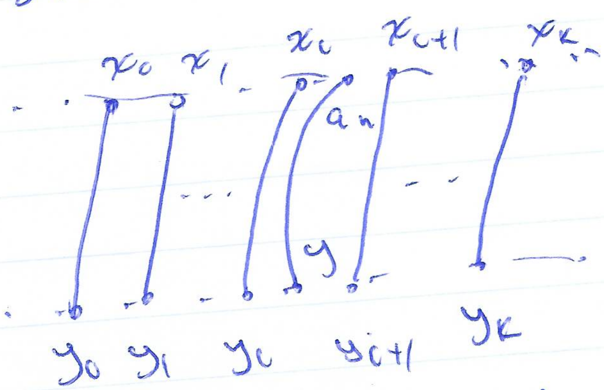
possible since no top pt



case 2

in all cases: f_{2n+1} is a partial order preserving injection extending f_{2n} s.t. $b_n \in ran(f_{2n+1})$

Step 2n: Same idea: given f_{2n-1} define $f_{2n} = f_{2n-1}$ if $a_n \in dom(f_{2n-1})$ already, otherwise find $y \in |B|$ s.t. $f_{2n} = f_{2n-1} \cup \{(a_n, y)\}$ still order preserving.



then: f_{2n} is order preserving, extends f_{2n-1} and $a_n \in dom(f_{2n})$

Let $f = \bigcup_{k \in \omega} f_k$.

<u>Claim</u>: f is an isomorphism of A with B.

<u>Pf</u>: (i) f is injective since if $x, x' \in \mathrm{dom}(f)$ and $x \neq x'$ then we have $x, x' \in \mathrm{dom}(f_k)$ for some k hence by injectivity of f_k we have:

$$f(x) = f_k(x) \neq f_k(x') = f(x')$$

(ii) f is <u>total</u>: if $x \in |A|$ then $x = a_n$ for some n hence $x \in \mathrm{dom}(f_{2n}) \subseteq \mathrm{dom}(f)$

(iii) f is <u>surjective</u>. if $y \in |B|$ then $y = b_n$ for some n hence $y \in \mathrm{ran}(f_{2n+1}) \subseteq \mathrm{ran}(f)$

(iv) f is <u>order-preserving</u> since if $x < x'$ in $|A|$ then for some k we have

$$f(x) = f_k(x) < f_k(x') = f(x')$$

since f_k is order preserving.

✓

Proof can be adapted to prove following stronger fact:

<u>Thm</u>: if A, B are models of DLO and $S = \{x_0 < x_1 < \dots < x_n\}$ is a finite subset of A and
$T = \{y_0 < y_1 < \dots < y_n\}$ is a finite subset of B then there is an isomorphism $f: A \to B$ s.t. $f(x_i) = y_i$ for all $i \leq n$.

<u>Pf</u>: same: just begin with $f_0 = \{(x_0, y_0), \dots, (x_n, y_n)\}$

↳ Thm says: any finite <u>partial</u> isomorphism between two models of DLO can be extended to an isomorphism.

In particular:

<u>Corollary</u>: if $A \models DLO$ and $f_0: A \to A$ is a finite partial automorphism then f_0 can be extended to an automorphism $f: A \to A$.

→ This theorem also gives a lot of logical info about dense orders.

↳ A theory Σ is called countably categorical if any two ctbl models of Σ are isomorphic.

e.g. DLO is ctbly categorical

What else?
The empty theory $\Sigma = \emptyset$ (over the empty language) is too

Another ex. we've seen:
- let S be a unary relation symbol
- let φ_n be the sentence "there are at least n elements in S"
- let ψ_n be " " " " not in S"

Then $\Sigma = \{\varphi_n : n \in \omega\} \cup \{\psi_n : n \in \omega\}$ is ctbly categorical.

Theorem DLO is a complete theory

Pf: If not, there is a sentence φ s.t. DLO $\cup \{\varphi\}$ is consistent and DLO $\cup \{\neg\varphi\}$ is consistent.

Let A be a model of DLO $\cup \{\varphi\}$
Let B " " of DLO $\cup \{\neg\varphi\}$

Then A, B are infinite since in particular they model DLO.

Let A^* be a ctbl elem substructure of A
Let B^* " " " " of B

These exist by downward LS.

Then A^*, B^* also model DLO and are ctbl, hence isomorphic by our theorem.

But $A^* \models \varphi$, and $B^* \models \neg\varphi$ contradiction! Hence DLO is complete.

↳ the same proof works for **any** ctbly categorical theory Σ, as long as Σ has no finite models.

Another important result that follows from Cantor's theorem:

Theorem (Quantifier elimination for DLO). Suppose $\varphi(\bar{u})$ is a fmla w/ free variables among $\bar{u} = u_1, \ldots, u_n$.

its lang w/ just $<$

Then there is a quantifier free fmla $\psi(\bar{u})$ s.t.

$$DLO \vdash \forall \bar{u}\,(\varphi(\bar{u}) \Leftrightarrow \psi(\bar{u}))$$

or equivalently

$$DLO \models \forall \bar{u}\,(\varphi(\bar{u}) \Leftrightarrow \psi(\bar{u}))$$

→ theorem says that any assertion (i.e. fmla) you can ~~Do~~ make about a finite set of points in a dense linear order is equivalent to a quantifier free assertion

Below I write $<$ instead of R.

e.g. - consider the fmla $\varphi(u_1, u_2)$:

$$\exists v\, (u_1 < v \wedge v < u_2)$$

- in a dense linear order this is equiv to $\psi(u_1, u_2)$:

$$u_1 < u_2.$$

- But in a general linear order the two assertions are not equivalent
- e.g. in $A = (\omega, <)$ we have

$$A \models \neg \varphi(1,2)$$

but $A \models \psi(1,2)$

Proof of theorem:

- let A be a ctbl model of DLO.
- Suppose $\bar{x} = x_1, \ldots, x_n \in |A|$
- Consider all formulas of the form $u_i < u_j$ and $\neg(u_i < u_j)$ for $i, j \leq n$.
- Let type$(\bar{x})$ be the set of all such formulas which are true about $x_1, \ldots, x_n$ in A

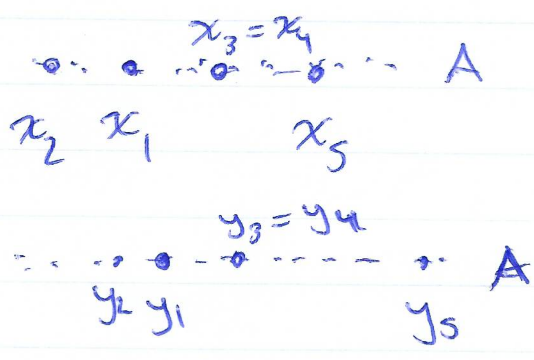
- e.g. if $x_1 < x_5$ is true in A then $u_1 < u_5 \in type(\bar{x})$
- if $x_3 = x_4$ then $\neg(u_3 < u_4) \in type(\bar{x})$

- observe $type(\bar{x})$ is a finite set

- Now: suppose $\bar{y} = y_1, \dots, y_n$ is another tuple of elements of $|A|$.

- Then if $type(\bar{y}) = type(\bar{x})$ the points $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are configured in the same way w.r.t. one another.

<u>Pic</u>:



$type(\bar{x}) = type(\bar{y})$

- Consider the partial map that sends $x_i \to y_i$ for $i \le n$.

- Then this map is order-preserving hence by our theorem before can be extended to an automorphism $\pi : A \to A$.

- Hence by Lemma 4.1 we have that for any fmla $\chi(u_1, \ldots, u_n)$

$A \models \chi(x_1, \ldots, x_n)$ iff
$A \models \chi(y_1, \ldots, y_n)$

- This says that whether or not $\chi(\bar{x})$ is true in A depends only on configuration of the points $x_1, \ldots, x_n$ in A
  i.e. only on $\text{type}(\bar{x})$.

- Now, ~~[illegible]~~ there are only finitely many configurations of n-many points
  i.e. $\{\text{type}(\bar{x}) : x_1, \ldots, x_n \in |A|\}$
  is finite

- In particular, if we let $T = \{\text{type}(\bar{x}) : A \models \varphi(\bar{x})\}$ then T is finite

- say $T = \{\tau_1, \dots, \tau_k\}$

- for each $i \le k$ let $\delta_i(\bar{u})$ be conjunction of fmlas in τ_i

- let $\psi(\bar{u})$ be the disjunction $\delta_1(\bar{u}) \vee \dots \vee \delta_k(\bar{u})$

<u>Then</u>: for any tuple $\bar{x} = x_1, \dots, x_n \in |A|$ we have

$A \models \varphi(\bar{x})$

iff

$\text{type}(\bar{x}) \in T$, i.e. $\text{type}(\bar{x}) = \tau_i$ for some $i \le k$

iff

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$A \models \tau_i(\bar{x})$ for some $i \leq k$

iff

$A \models \delta_i(\bar{x})$ for some $i \leq k$

iff

$A \models \delta_1(\bar{x}) \vee \dots \vee \delta_k(\bar{x})$

i.e. iff

$A \models \psi(\bar{x})$. quantifier free

But this shows

$A \models \forall \bar{u} \, (\varphi(\bar{u}) \Leftrightarrow \psi(\bar{u}))$!

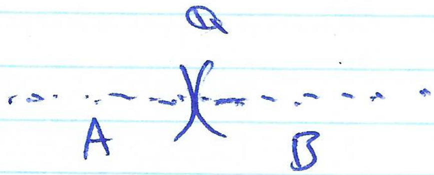
(i)

- → can also use our characterization of $(\mathbb{Q}, <)$ as unique (up to isomorphism) ctbl model of DLO to characterize $(\mathbb{R}, <)$

- For those who have never seen a construction of $\mathbb{R}$, here is a sketch.

- $\mathbb{R}$ is obtained by "filling in the holes in $\mathbb{Q}$"

- Sps we partition $\mathbb{Q}$ into a left interval A and right interval B ($A \cap B = \emptyset$)



- such a partition is called a <u>cut</u>. We denote it (A, B)

- there are three (exclusive) possibilities:

  (1) A has a top point
  (2) B has a bottom point
  (3) neither (1) nor (2)

e.g. (1) $A = \{q \in \mathbb{Q} : q \leq 0\}$ $B = \{q \in \mathbb{Q} : q > 0\}$
(2) $A = \{q \in \mathbb{Q} : q < 0\}$ $B = \{q \in \mathbb{Q} : q \geq 0\}$
(3) $A = \{q \in \mathbb{Q} : q^2 < 2\}$ $B = \{q \in \mathbb{Q} : q^2 > 2\}$

(ii)

- A partition of type (3) is called a **gap**

- $\mathbb{R}$ is obtained by filling every gap in $\mathbb{Q}$ with a single point. ∎

- one can then show
  (1) $(\mathbb{R}, <)$ has least upper bound property
  (2) $\mathbb{Q}$ is dense in $\mathbb{R}$ (i.e. between any two reals there is a rational).

- these two properties characterize $(\mathbb{R}, <)$ in the following strong sense

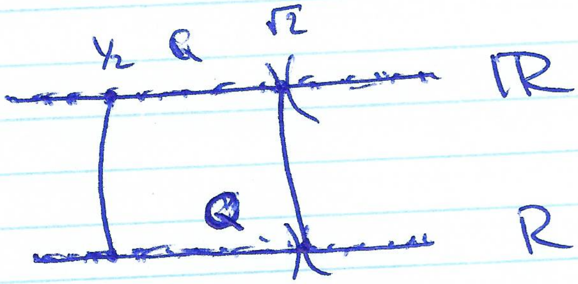
Theorem. Suppose $(R, <)$ is a model of DLO with the l.u.b. property.
Suppose further that $(R, <)$ contains a ctbl elementary substructure $(Q, <)$ s.t. Q is dense in R.
Let $\pi : \mathbb{Q} \to Q$ be an isomorphism.

Then $(R, <)$ is isomorphic to $(\mathbb{R}, <)$

Moreover: there is an isomorphism $\sigma : \mathbb{R} \to R$ s.t. $\sigma \upharpoonright \mathbb{Q} = \pi$.

(iii)

Proof by picture:



i.e. σ is defined by:

if $x \in \mathbb{R}$ then there is a cut (A, B) s.t. x is l.u.b. of A

let $\sigma(x) = y$ where y is l.u.b.

$\pi[A] = \{\pi(q) : q \in A\}$ ✓

(i) Nonstandard Analysis

- back in early days of calculus ppl wanted to reason about "infinitesimals," i.e. numbers smaller than every positive real number but still greater than 0

- Leibniz used dx, dy etc. to denote infinitesimals which we still use today... but is only "notational" — no substantial meaning in itself.

- at time, couldn't put infinitesimals on rigorous ground, so theory was supplanted by theory of limits.

- but infinitesimals can be put on rigorous ground... with model theory (1960s)!

↳ goal: Find a structure $\mathbb{R}^*$, which contains (a copy of) $\mathbb{R}$ as an elementary substructure, but also contains infinitesimal (and infinite) numbers.

(ii)

→ Consider a structure A with $|A| = \mathbb{R}$ and some of the usual constants, relations, functions

- e.g. could take
$$A = (|A|, R^A) = (\mathbb{R}, <)$$

- or
$$A = (|A|, c^A, d^A, f^A, g^A, h^A, i^A, j^A, R^A)$$
$$= (\mathbb{R}, 0, 1, +, -, \times, \div, |\cdot|, <)$$
absolute value

- this is version of $\mathbb{R}$ used on first HW prob.

- since $\div$ is not total on $\mathbb{R}$, we extend it to a total function

- doesn't matter how we do this, e.g. could define $x \div 0 = 0$ for all $x \in \mathbb{R}$.

- Could also throw in other functions relations constants if we like

(iii)

- <u>Now</u>: recall that
  $\mathrm{Diag}(A) =$ all sentences true of A in expanded lang w/ constants
  $= \{\varphi : (A, x)_{x \in \mathbb{R}} \models \varphi\}$

- Previously we used compactness to construct elementary extension of $(\omega, <)$ with "infinite elements"

$$\circ \quad \circ \quad \circ \quad \cdots \qquad \cdots \bullet \cdots$$

- Following same procedure can get elementary extension of (a copy, of) A, with infinite and infinitesimal elements.

- Let d be a new constant symbol. We consider the theory
  $\mathrm{Diag}(A) \cup \underbrace{\{c_x < d : x \in \mathbb{R}\}}_{= |A|}$

- For every finite subset $\Delta = \Delta_0 \cup \Delta_1$ of this theory can get a model for Δ

(iv)

by simply taking $(A, x)_{x \in \mathbb{R}}$ and interpreting d as some $r \in \mathbb{R}$ greater than all x's s.t. c_x appears in Δ,

- Hence by compactness there is a model E of $\mathrm{Diag}(A) \cup \{c_x < d : x \in \mathbb{R}\}$

- Let B be reduct of E to orig. lang. of A.

- By Lemma 4.7 we knew B contains an elem. substructure A^* which is isomorphic to A where $|A^*| = \{c_x^B : x \in \mathbb{R}\}$ and isomorphism is $\pi : A \to A^*$ defined by $\pi(x) = c_x^B$

- We could replace A^* with A to obtain a model in which A is <u>literally</u> an elementary substructure if we like

  but book and HW probs don't do this.

(V)

- But still, we think of A^* "as" A.

- And we have $A^* \leq B$.

- Hence if $\varphi(u_1, \dots, u_n)$ is a fmla and $x_1, \dots, x_n \in |A| = \mathbb{R}$ we have

$$A \models \varphi(x_1, \dots, x_n)$$

iff $A^* \models \varphi(c^B_{x_1}, \dots, c^B_{x_n})$ ← since π is isomorphism

$\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{\pi(x_1)}, \dots, \overset{\shortparallel}{\pi(x_n)}$

iff $B \models \varphi(c^B_{x_1}, \dots, c^B_{x_n})$ ← since $A^* \leq B$.

- But while B contains an elementary copy of A, B is not ~~[illegible]~~ isomorphic to A

- in particular B contains the "infinite element" d^B which is larger than every elt of A^*

(vi)

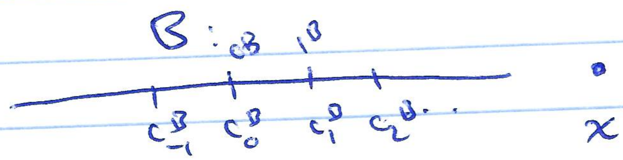
- Let $\text{Finite}(\mathcal{B}) = \{ y \in |\mathcal{B}| :$ there are $x, z \in |A^*|$ s.t. $x <^{\mathcal{B}} y <^{\mathcal{B}} z \}$

- Let $\text{Infinite}(\mathcal{B}) = |\mathcal{B}| - \text{Finite}(\mathcal{B})$

- Observe $A^* \subseteq \text{Finite}(\mathcal{B})$
  $d^{\mathcal{B}} \in \text{Infinite}(\mathcal{B})$.

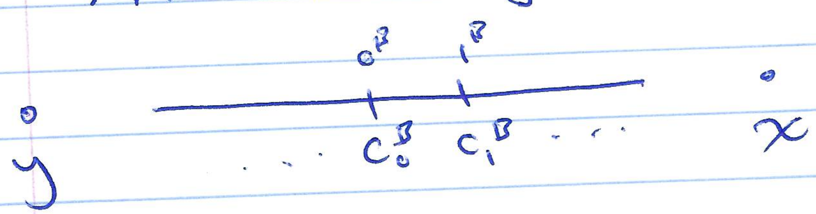
- For the moment let $x = d^{\mathcal{B}}$

Picture:



- Now: the sentence $\forall u \exists v \, (u + v = 0)$ is true in A, hence in A^* and $\mathcal{B}$.

- Hence there must be some $y \in |\mathcal{B}|$ s.t. $x +^{\mathcal{B}} y = 0^{\mathcal{B}}$



(vii)

- We must have $y <^B 0^B$
Since $B \models \forall u\,(0<u \Rightarrow \exists v\,(v<0 \wedge u+v=0))$
$\wedge \forall w\,(u+w=0 \Rightarrow v=w))$

- Can also show we must have
$y \in Infinite(B)$ (try it!)

- Can also show: for any $r \in Finite(B)$
we have $x+r$ infinite

- Hence B contains infinitely many infinite elements
↳ positive as well as negative.

- But B also contains infinitesimal elements!

- This is because the sentence
$\forall u \forall v\,((1<u \wedge u<v) \Rightarrow 1 \div u < 1 \div v)$
is true in A, hence in A^* and B

and so is:
$\forall w\,(w \neq 0 \Rightarrow 1 \div (1 \div w) = w)$

(viii)

- Hence if we let
$$\varepsilon = 1^{\mathbb{B}} \div^{\mathbb{B}} x$$
we must have
$$0^{\mathbb{B}} <^{\mathbb{B}} \varepsilon <^{\mathbb{B}} r$$
for every $r \in |A^*|$

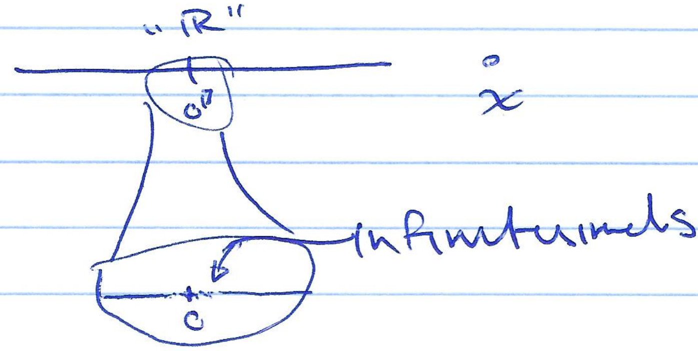
- So ε is a positive number that is less than every positive "real" number (i.e. every positive member of A^*)

- We call ε an infinitesimal.

- More generally we define $\delta \in |\mathbb{B}|$ to be infinitesimal iff

$$0^{\mathbb{B}} <^{\mathbb{B}} |\delta| <^{\mathbb{B}} r \quad \text{for every } r \in |A^*|$$
with $0^{\mathbb{B}} < r$.

$\mathbb{B}$:



(i) # More Nonstandard Analysis

<u>From last time:</u>

- Our version of $\mathbb{R}$ was:

$$A = (\mathbb{R}, 0, 1, +, -, \times, \div, |\cdot|, <)$$

Though we could've enriched this starting structure if we liked.

- Indeed we could have equipped A with <u>every</u> constant, relation, and function on $\mathbb{R}$ (this is usually what's done)

- For today let's just assume we have two unary relation symbols that we interpret as $\mathbb{Z}$ and $\mathbb{Q}$. (i.e. we have S, R s.t. $S^A = \mathbb{Z}$, $R^A = \mathbb{Q}$)

- Let's also assume we have constant symbols for every $x \in \mathbb{R}$ in our base language

(ii)

- We write:

$$A = (\mathbb{R}, \{c_x\}_{x\in\mathbb{R}}, +, -, \times, \div, |\cdot|, <, \mathbb{Z}, \mathbb{Q})$$

where $c_x^A = x$ for all $x \in \mathbb{R}$

- Then we Construct $B \models \mathrm{Diag}(A) \cup \{c_x < d : x \in \mathbb{R}\}$ where d is new constant symbol.

- I'll write:

$$B = (|B|, d^B, \{c_x^B\}_{x\in\mathbb{R}}, +^B, -^B, \times^B, \div^B, |\cdot|^B, <^B, \mathbb{Z}^B, \mathbb{Q}^B)$$

- Abusing notation a lot here: Really we have constant, function, relation symbols that in A are interpreted as the elements of $\mathbb{R}$, $+, -, \ldots$ etc. and in B are interpreted as new constants, relations, functions on $|B|$.

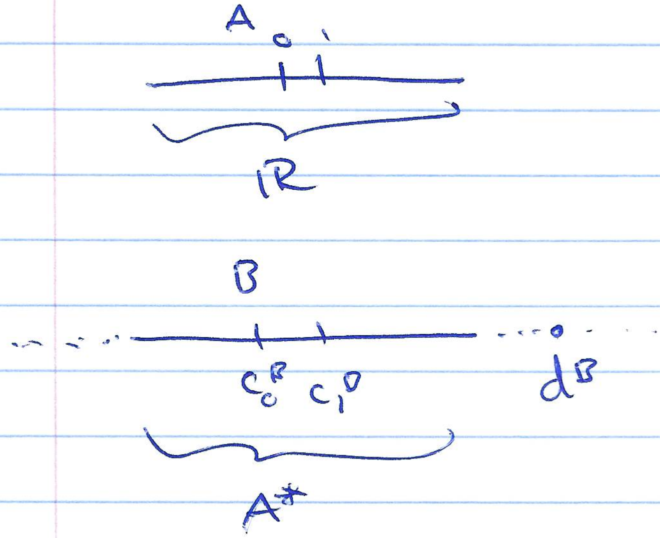
- We know there is $A^* \subseteq B$ with A^* isomorphic to the reals A

- Iso is $\pi : A \to A^*$
$$\pi(x) = c_x^B$$

(iii)

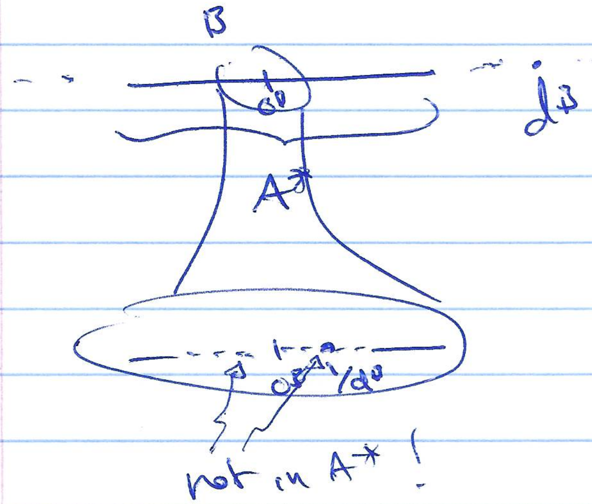
- d^B is larger than all $x \in |A^*|$

Picture:



I'll write 0^B, 1^B, etc. as shorthand for c_0^B, c_1^B, etc.

- we have $1^B/d^B$ is infinitesimal in B, i.e. strictly positive but smaller than every $r \in A^*$



(iv)

- We defined

$$\text{Finite}(\mathbb{B}) = \{x \in \mathbb{B} : \text{there is } r, s \in |A^*| \; r <^{\mathbb{B}} x <^{\mathbb{B}} s\}$$

$$\text{Infinite}(\mathbb{B}) = |\mathbb{B}| - \text{Finite } \mathbb{B}$$

- We have

$$A^* \subseteq \text{Finite}(\mathbb{B})$$
$$d^{\mathbb{B}} \in \text{Infinite}(\mathbb{B})$$

- We defined $\delta \in |\mathbb{B}|$ to be <u>infinitesimal</u> iff $0 < |\delta|^{\mathbb{B}} < r$ for every $r \in A^*$

- Same proof that $1^{\mathbb{B}}/d^{\mathbb{B}}$ is infinitesimal proves $1^{\mathbb{B}}/x$ is infinitesimal for x infinite.

Some more practice proving facts about $\mathbb{B}$:

<u>Facts</u>:

(i) if $x, y \in \text{Finite}(\mathbb{B})$ then $x +^{\mathbb{B}} y \in \text{Finite}(\mathbb{B})$ and $x \times^{\mathbb{B}} y \in \text{Finite}(\mathbb{B})$

(v) 71

(ii) if δ is infinitesimal then $1/\delta$ is in Infinite(B)

(iii) if $x \in$ Finite(B) and δ infinitesimal then $\delta \times^B x$ is infinitesimal

(iv) $\mathbb{Q}^B$ is dense in $|B|$, i.e. if $x, y \in |B|$ and $x <^B y$ then there is $z \in \mathbb{Q}^B$ $x <^B z <^B y$.

Pf (1) Fix $x, y \in$ Finite(B)

Then there is $r, s \in A^*$

$0^B -^B r^B \longrightarrow -r < x < r$

$-s < y < s$

We know:

$$A \models \forall u \forall v \big(\exists w \exists y (-w < u < w \wedge -y < v < y) \Rightarrow (-(w+y) < u+v < w+y \wedge -wy < u \times v < wy) \big)$$

hence A^* does

hence B does (by elementarity)

(2) Sps δ is infinitesimal and for convenience $\delta > 0^B$

(vi)

c_n^B

- Then $\delta < 1^B/n^B$ for all $n \in \omega$
- We know:

$$B \models \forall x\,(0 < x < 1/c_n \Rightarrow c_n < 1/x)$$

for all $n \in \omega$ since A models this statement

- hence $c_n^B = n^B < 1/\delta$ for all $n \in \omega$
- hence $1/\delta > r$ for all $r \in A^+$

i.e. $1/\delta$ is infinite

(3) Fix $x \in \text{Finite}(B)$, δ infinitesimal. Again for convenience assume x, δ both positive.

There is N s.t. $x < N^B$

We know that, for every $n \in \omega$,

$$B \models \forall u \forall v\,(0 < u < 1/c_n \wedge v < c_N \Rightarrow u \cdot v < c_n/c_N)$$

since A models this statement

Hence $x \times^B \delta < N^B/n^B$ for all $n \in \omega$.

Hence $x <^B r$ for all $r \in A^+$ (why?)
$r >^B 0^B$

(vii)

"$q \in \mathbb{Q}$"

(4) $\mathcal{B} \models \forall u \forall v \,(u < v \Rightarrow \exists q \,(\mathbb{Q}(q) \wedge u < q < v))$

Since $\mathcal{A}$ does. ✓

- Observe ① $\Rightarrow$ $r \in \text{Finite}(\mathcal{B})$ and $x \in \text{Infinite}(\mathcal{B})$ then $r + x \in \text{Infinite}(\mathcal{B})$

<u>Why</u>: if not then $r + x \in \text{Finite}(\mathcal{B})$ hence $r + x + (-r) = x \in \text{Finite}(\mathcal{B})$ by (1), contradiction. Why?

- In proving ①–④ repeatedly used that $\mathcal{B}$ contains a copy $\mathcal{A}^*$ of the reals $\mathcal{A}$ as an <u>elementary</u> substructure

- elementary substructures have the same <u>first-order</u> properties as the structures they live in

⟶ <u>NOT</u> necessarily the same second order properties

(vii)

- e.g. it follows from (4) and some stuff on HW that $\mathbb{Q}^{\mathcal{B}}$ is <u>uncountable</u>

- We also have the following:

<u>Prop'n</u>: $\mathcal{B}$ does <u>not</u> have least upper bound property.

<u>Pf</u>: - We prove $|A^*|$ does not have an l.u.b. in $\mathcal{B}$.
  - Toward a contradiction suppose x is l.u.b. for $|A^*|$. Then $x \in$ Infinite($\mathcal{B}$)
  - Consider $x -^{\mathcal{B}} 1^{\mathcal{B}}$.
  - Must have $x -^{\mathcal{B}} 1^{\mathcal{B}} \in$ Finite($\mathcal{B}$) since x is l.u.b. for A^*. and $x-1 < x$. (why?)
  - So there is $r, r' \in |A^*|$ s.t. $r < x-1 < r'$
  - But then $r+1 < x < r'+1$ (why) i.e. x is finite
  - contradiction ✓