Ch. 4 : Intro to Model Theory

- "Model Theory" = study of models of first-order theories.

- Some useful notation (now that we're done w/ formal deduction)
  - if φ a fmla w/ free variables among $u_1, \ldots, u_n$ we write $\varphi(u_1, \ldots, u_n)$ or just $\varphi(\bar{u})$
  - if t is a (not necessarily closed) term w/ free variables among $u_1, \ldots, u_n$ we ~~may~~ write $t(u_1, \ldots, u_n)$ or $t(\bar{u})$.

Ex: · If φ is $\exists u (u \approx v)$ will usually write $\varphi(v)$, but could write $\varphi(v, w)$, but not $\varphi(u)$ or $\varphi(x, y)$.
  - If t is $f(u, v)$ could write $t(u, v)$ or $t(u, v, w)$ but not $t(u, w)$.

- Will also simplify notation for "subbing elements of $|A|$ into φ"
  - if $\varphi(u_1, \ldots, u_n)$ is fmla and A is a structure and $x_1, \ldots, x_n \in A$ we write

$$A \models \varphi(x_1, \ldots, x_n)$$

as shorthand for

$$(A, x_1, \ldots, x_n) \models \varphi(u_1/c_{x_1})(u_2/c_{x_2}) \cdots (u_n/c_{x_n})$$

Ex: if $\varphi(u,v)$ is $R(u,v)$ and $A = (|A|, R^A) = (\mathbb{Z}, <)$ then we'll write:

$$A \models \varphi(1,2) \text{ or even } A \models 1<2$$
and $A \not\models \varphi(4,2)$ or even $A \not\models 4<2$

An important fact you guys proved on HW is: "isomorphisms are elementary embeddings":

Lemma 4.1: - Let $\pi : A \to B$ be an isomorphism
- If $\varphi(u_1, \ldots, u_n)$ a fmla and $x_1, \ldots, x_n \in |A|$ we have:

$$A \models \varphi(x_1, \ldots, x_n) \text{ iff } B \models \varphi(\pi(x_1), \ldots, \pi(x_n))$$

Pf: (exercise 3.5)

Ex: Let $A = B = (\mathbb{Z}, <)$
Let $\pi : \mathbb{Z} \to \mathbb{Z}$ be defined by $\pi(x) = x+1$ (you guys showed this defines automorphism of $\mathbb{Z}$)

Lemma says that, e.g., for any fmla $\varphi(u,v)$ we have

$$(\mathbb{Z}, <) \models \varphi(1,3) \text{ iff}$$
$$(\mathbb{Z}, <) \models \varphi(2,4)$$

Substructures

- "substructures" are just subsets of the universe closed under all functions and constants

ubiquitous notion: "subgroup" "subset" "subspace" all instances of this concept

Def'n: Let A, B be structures in the same language.

A is a substructure of B if:

- $|A| \subseteq |B|$
- $c^A = c^B$ for all constants c in lang.
- $F^A = F^B \restriction |A|^n$ for all n-ary functions F in lang.
- $R^A = R^B \cap |B|^n$ for all n-ary relations symbols R in lang.

Ex - let our lang. be $\{c, R, f\}$ where R is binary, f is unary.

- let $B = (|B|, c^B, R^B, f^B)$
$= (\mathbb{Z}, 0, <, d)$
where d is the double successor function $d(n) = n+2$.
- Let E denote the set of even integers
- Let $A = (|A|, c^A, R^A, f^A)$
$= (E, 0, <, d)$

Then A is a substructure of B.

Note: - There is no substructure $\mathcal{A}$ of $\mathcal{B}$ with $|\mathcal{A}|$ = set of odds (because $c^{\mathcal{B}} = 0$ not contained)

- There is also no substructure $\mathcal{A}$ with $|\mathcal{A}| = \{\ldots, -3, 0, 3, 6, \ldots\}$ (because this set is not closed under $f^{\mathcal{B}} = d$)

Generally: can always restrict relations to arbitrary subsets w/o issue, but w/ functions and constants need closure.

Lemma 4.2 - Let $\mathcal{B}$ be an infinite structure in a countable language.

- Suppose X is a ctbl subset of $|\mathcal{B}|$ (possibly finite)

Then there is a countable substructure $\mathcal{A}$ of $\mathcal{B}$ w/ $X \subseteq |\mathcal{A}|$

Pf: idea: begin with X and close under constants and functions to form $|\mathcal{A}|$

→ first let $Y = X \cup \{c^{\mathcal{B}} \mid c \text{ in lang}\}$ (Y is still ctbl)

→ Define a sequence of subsets $Y = Z_0 \subseteq Z_1 \subseteq \cdots$ inductively.

$\rightarrow Z_{i+1} = Z_i \cup \{F^B(x_1, \ldots, x_n) : F$ is n-ary function symbol and $x_1, \ldots, x_n \in Z_i\}$

(Z_{i+1} is still ctbl if Z_i is)
(Since Z_0 ctbl all Z_i are)

Let $Z = \bigcup_{i \in \omega} Z_i$ (still ctbl)

Notice: For every function symbol F, Z is closed under F^B

<u>Why</u>: given $x_1, \ldots, x_n \in Z$ there is i s.t. $x_1, \ldots, x_n \in Z_i$ hence $F^B(x_1, \ldots, x_n) \in Z_{i+1}$

Now define A:

- $|A| = Z$
- $c^A = c^B$ for all c (all these in $|A|$)
- $R^A = R^B \restriction Z^n$ for all R
- $F^A = F^B \restriction Z^n$ for all n-ary F (works by construction)

Then A is evidently a substructure of B that is ctbl and $X \subseteq |A|$. ✓

$\hookrightarrow$ Lemma actually constructs <u>minimal</u> substructure of B containing X.

Ex: - Suppose $\mathcal{B} = (|B|, c^B, R^B, f^B)$
$= (\mathbb{Z}, 0, <, d)$ as before.
- Let $X = \{2\}$
- Then A given by pf. of Lemma is $(E, 0, <, d)$

- OTOH if $X = \{1\}$ then A given is all of B. (Since must contain 0, 1 and hence all evens and odds by closure under d).

Lemma 4.3: - Suppose A is a substructure of B. and $\varphi(u_1, \dots, u_n)$ is a formula with no quantifiers
- Then if $x_1, \dots, x_n \in A$ we have
$A \models \varphi(x_1, \dots, x_n)$ iff
$B \models \varphi(x_1, \dots, x_n)$

"A thinks the same things are true that B does... for the elements of $|A|$"

Pf (sketch): by induction on comp. of φ.

Subclaim: if $t(u_1, \dots, u_n)$ a term and $x_1, \dots, x_n \in |A|$ then $t^A(x_1, \dots, x_n) = t^B(x_1, \dots, x_n)$
Pf: if t is c true since A is a substructure
if t is $f(t_1, \dots, t_n)$ then true by induction ✓

- if $\varphi^{(x_1,\dots,x_n)}$ is $s(x_1,\dots,x_n) = t(x_1,\dots,x_n)$ then lemma is true by subclaim
- if $\varphi^{(x_1,\dots,x_n)}$ is $R(x_1,\dots,x_n)$ then lemma is true since A a substructure.
- if lemma true for φ, ψ then true for $\neg\varphi$, $\varphi \wedge \psi$.

- no quantifier case ✓

Ex Suppose

$$A = (|A|, c^A, d^A, F^A, G^A, R^A) = (\mathbb{Q}, 0, 1, +, \times, <)$$

$$B = (\mathbb{R}, 0, 1, +, \times, <)$$

Then A is a substrure of B.

Suppose $\varphi(u)$ is $F(d,d) \approx u$
(i.e. $1+1=u$)

Then $A \models \varphi(2)$ and $B \models \varphi(2)$
$A \not\models \varphi(3)$ and $B \not\models \varphi(3)$

$B \not\models \varphi(\sqrt{2})$ but $\varphi(\sqrt{2})$ is neither true nor false in A since $\sqrt{2} \notin |A| = \mathbb{Q}$.

Ex Let χ be the sentence
$\exists v (G(v,v) \approx F(d,d))$
i.e $\exists v (v \times v = 1+1)$

Then ψ has no free variables but has a quantifier

We have $B \models \psi$
but $A \not\models \psi$

This example shows Lemma 4.3 cannot be extended to fmlas w/ quantifiers

Def'n: We say that A is an *elementary* substructure of B, and write $A \preceq B$, iff A is a substructure of B and for all $\varphi(u_1, \dots, u_n)$ and $x_1, \dots, x_n \in |A|$ we have

$$A \models \varphi(x_1, \dots, x_n) \quad \text{iff}$$
$$B \models \varphi(x_1, \dots, x_n$$

So our last example show that $(\mathbb{Q}, 0, 1, \times, +, <)$ is not elementary in $(\mathbb{R}, 0, 1, \times, +, <)$

↳ being an elem. substructure much stronger than being a substructure:

- by 4.3 a quantifier free fmla φ is true in a substructure A iff true in structure B
- for elem substructures A, φ is true in A iff true in B.

↳ Lemma 4.2 says we can always find ctbl substructures. What about ctbl elementary substructures?

↳ following gives us a criterion that will allow us to build elem substructures

Tarski-Vaught test

Sps A is a substructure of B
Assume for every fmla $\varphi(u_1, \dots, u_n, v)$ with n+1 free variables and every $x_1, \dots, x_n \in |A|$ we have:

$$\text{if } B \models \exists v\, \varphi(x_1, \dots, x_n, v)$$
$$\text{then there is } y \in |A| \text{ s.t.} \qquad (*)$$
$$B \models \varphi(x_1, \dots, x_n, y)$$

Then A is an elementary substructure of B.

<u>Pf</u>: - Sps (*) holds.
- We attempt to repeat the induction from lemma 4.3 to prove

<u>Claim</u> for <u>every</u> fmla $\varphi(\bar{u})$ and $\bar{x} \in |A|$ we have $A \models \varphi(\bar{x})$ iff $B \models \varphi(\bar{x})$

(this just says A is elementary in B)

<u>Pf</u>: - exactly the same as in 4.3, except now in induction step need to consider quantified statements
- $\exists$ statements enough.

(IH) Sps we have the claim for $\varphi(\bar{u}, v)$ and our formula is $\exists v \varphi(\bar{u}, v)$

(IS) Fix $\bar{x} \in |A|$.

We WTS: $A \models \exists v \varphi(\bar{x}, v)$ iff $B \models \exists v \varphi(\bar{x}, v)$

($\Rightarrow$) Sps $A \models \exists v \varphi(\bar{x}, v)$
Then there is $y \in |A|$ s.t. $A \models \varphi(\bar{x}, y)$
By IH $B \models \varphi(\bar{x}, y)$
Hence $B \models \exists v \varphi(\bar{x}, v)$

($\Leftarrow$) Sps $B \models \exists v \varphi(\bar{x}, v)$.
Then by * there is $y \in |A|$ s.t. $B \models \varphi(\bar{x}, y)$
Hence by IH
$A \models \varphi(\bar{x}, y)$
Hence $A \models \exists v \varphi(\bar{x}, v)$

↳ in general, not easy to prove a given substructure A is elementary in $\mathcal{B}$

↳ can show that ctbl elementary substructures always exist

→ close under constants, functions, relations and find witnesses for all existential fmlas.

Downward Löwenheim–Skolem Theorem

Let $\mathcal{B}$ be a structure in a ctbl lang.
Let $X \subseteq |\mathcal{B}|$ be a ctbl subset.
There exists an elementary substructure $\mathcal{A}$ of $\mathcal{B}$ with $\mathcal{A}$ ctbl and $X \subseteq |\mathcal{A}|$

Pf. - We may assume $\mathcal{B}$ is infinite (if not, $\mathcal{B} = \mathcal{A}$ works)

- For every fmla $\varphi(u_1, \dots, u_n, v)$ w/ $n+1$ free variables we define a function

$$f_\varphi : |\mathcal{B}|^n \to \mathcal{B}$$

as follows:

Given $x_1, \dots, x_n \in \mathcal{B}$, if
$\mathcal{B} \models \exists v\, \varphi(x_1, \dots, x_n, v)$
then pick $y \in |\mathcal{B}|$ s.t.
$\mathcal{B} \models \varphi(x_1, \dots, x_n, y)$
and let $f_\varphi(x_1, \dots, x_n) = y$.

OTOH, if
$\mathcal{B} \not\models \exists v\, \varphi(x_1, \dots, x_n, v)$
then pick any $z \in |\mathcal{B}|$ and let
$f_\varphi(x_1, \dots, x_n) = z$

- (we use AC here)

- So for every $x_1, \dots, x_n \in |B|$ $f_\varphi(x_1, \dots, x_n)$ is a witness to $\exists v\, \varphi(x_1, \dots, x_n, v)$ ... if such a witness exists.

- Notice: only ctbly many such f_φ (lang is ctbl)

- Now, for every φ introduce a new function symbol F_φ.

- Define an expanded structure B^* that interprets each F_φ as f_φ, i.e. $F_\varphi^{B^*} = f_\varphi$.

same universe etc. as B.

- By lemma 4.2, B^* has a ctbl substructure A^* s.t. $X \subseteq |A^*|$

- Now: let A be the reduction of A^* to the original lang. w/o new function symbols

- Then A is (clearly) a substructure of B

- Observe: if $\varphi(\bar{u}, v)$ and $\bar{x} \in |A| = |A^*|$ then $f_\varphi(\bar{x}) = F_\varphi^{A^*}(\bar{x}) = y$ is in $|A|$ as well.

- Hence if $\mathcal{B} \models \exists v \varphi(\bar{x}, v)$
  then $\mathcal{B} \models \varphi(\bar{x}, y)$ point w. in $|A|$!

- And the same w true for all $\varphi(\bar{u}, v)$
i.e. A contains witness to all existential fmlas

- By Tarski-Vaught, A is elementary in $\mathcal{B}$ ✓

<u>Ex</u>: - want to give an example of an elem. substructure.
- Let $\mathcal{B} = (|\mathcal{B}|, R^{\mathcal{B}}) = (\mathbb{R}, <)$
- Then $A = (\{0,1\}, <)$
is a substructure of $\mathcal{B}$
- but not elementary:

$$\mathcal{B} \models \exists u((0<u) \wedge (u<1))$$
$$A \not\models \exists u(0<u \wedge (u<1)$$

However, you will show on Homework that
$A = (\mathbb{Q}, <)$
is elementary in $\mathcal{B}$!