A word on quantifiers

Some useful notation: will write $\varphi(v)$ to mean φ is a fmla w/ a single free variable v, $\psi(u,v)$ means ψ has two free variables u and v etc.

Recall: • $A \models \exists v\, \varphi(v)$ iff there is $x \in |A|$ s.t. $(A,x) \models \varphi(v/c_x)$
• $A \models \forall v\, \varphi(v)$ iff for every $x \in |A|$, we have $(A,x) \models \varphi(v/c_x)$

But really: • $A \models \exists v\, \varphi(v)$ iff there is $x \in |A|$ s.t. "$\varphi(x)$ is true in A"
• $A \models \forall v\, \varphi(v)$ iff for every $x \in |A|$ "$\varphi(x)$ is true in A"

- have not defined what it means to "plug in" an arbitrary $x \in |A|$ for v; have defined $(A,x) \models \varphi(v/c_x)$

- but when working in a specific structure can often cut out consideration of extra constants e.g. Sps $L = \langle R \rangle$ is a lang w/ a binary relation symbol and $\mathcal{A} = \langle |\mathcal{A}|, R^{\mathcal{A}} \rangle$ is a structure in this lang.

Sps φ is $\forall u \exists v\, R(u,v)$

- Then $\mathcal{A} \models \varphi$ iff for every $x \in |\mathcal{A}|$ we have $(\mathcal{A}, x) \models \exists v\, R(c_x, v)$
- iff for every $x \in |\mathcal{A}|$ there is $y \in |\mathcal{A}|$ s.t. $((\mathcal{A}, x), y) \models R(c_x, c_y)$
- iff for every $x \in |\mathcal{A}|$ there is $y \in |\mathcal{A}|$ s.t. $R^{((\mathcal{A},x),y)}\left(c_x^{((\mathcal{A},x),y)}, c_y^{((\mathcal{A},x),y)}\right)$

iff $R^{\mathcal{A}}(x, y)$

→ on HW or exam can cut out middle lines and say:

$\mathcal{A} \models \forall u \exists v\, R(u,v)$

iff for every $x \in |\mathcal{A}|$ there is $y \in |\mathcal{A}|$ s.t. $R^{\mathcal{A}}(x,y)$

- e.g. if $|\mathcal{A}| = \mathbb{Z}$ and $R^{\mathcal{A}}$ is $<$ then

$\mathcal{A} \models \forall u \exists v\, R(u,v)$ iff for every $n \in \mathbb{Z}$ there is $m \in \mathbb{Z}$ s.t. $n < m$ (true)

More on expanding/reducing the language

- Suppose L_0, L_1 are languages and every symbol in L_0 is in L_1. We write $L_0 \subseteq L_1$.

- Sps A is a structure in L_0
  A' is a structure in L_1

  and $|A| = |A'|$

  and for every $c, f, R \in L_0$

  $$c^A = c^{A'}$$
  $$f^A = f^{A'}$$
  $$R^A = R^{A'}$$

  A, A' interpret symbols in L_0 the same

  then we have

  Fact: if φ is a sentence written in L_0 then

  $$A \models \varphi \text{ iff } A' \models \varphi.$$

  Pf: - induct on construction of terms and formulas written in L_0
  - main pt, if s is a term written in L_0 then $s^A = s^{A'}$.

ex: Sps $A = \langle \mathbb{R}, 0^A, 1^A, +^A, \leq^A \rangle$
$A' = \langle \mathbb{R}, 0^{A'}, 1^{A'}, 2^{A'}, +^{A'}, \leq^{A'} \rangle$

where all interps usual.

Then: $A \models 0 \leq 1$ and $A' \models 0 \leq 1$
$A \not\models 1 \leq 0$ and $A' \not\models 1 \leq 0$
$A' \models 0 \leq 2$, but A takes no stance

In particular: if $\mathcal{A}$ a struct, $x \in |\mathcal{A}|$, and φ is written in language of $\mathcal{A}$, then

i.e. φ does not have new symbol c_x

$$(\mathcal{A}, x) \models \varphi \text{ iff } \mathcal{A} \models \varphi.$$

Need this for next proof.

Specialization Lemma: Suppose t is a closed term and φ is a fmla w/ a single free variable v, then:

$$\forall v \varphi \models \varphi(v/t)$$

e.g.: if language is $\langle 0, 1, +, \leq \rangle$, t is $1+1$ and φ is $v \leq v+v$ then Lemma gives:

$$\forall v (v \leq v+v) \models (1+1) \leq (1+1)+(1+1)$$

Pf of Lemma: - Fix $\mathcal{A}$ and suppose $\mathcal{A} \models \forall v \varphi$ (WTS $\mathcal{A} \models \varphi(v/t)$)
- then for every $y \in |\mathcal{A}|$ we have $(\mathcal{A}, y) \models \varphi(v/c_y)$
- Suppose $t^{\mathcal{A}} = x \in |\mathcal{A}|$
- Then certainly
$$(\mathcal{A}, x) \models c_x \approx t$$

- Hence by substitution Lemma:

$(A, x) \models \varphi(v/c_x) \iff \varphi(v/t)$

- and since $(A, x) \models \varphi(v/c_x)$ must have $(A, x) \models \varphi(v/t)$.

- but $\varphi(v/t)$ is written in original language (w/o c_x) hence we have $A \models \varphi(v/t)$.

- Since A was arbitrary, lemma follows.

- Our next goal is to prove a kind of converse to above lemma (the generalization lemma)

- First we record some technical lemmas we will use in our proof of the completeness theorem for FOL.

<u>Lemma 3.4</u> ← numbering from textbook

Let A be a structure s.t. for every $x \in |A|$ there is a closed term t written in the language of A s.t. $t^A = x$. Let φ be a fmla w/ a single free variable. <u>Then</u> $A \models \forall v \varphi$ iff for every closed term t, $A \models \varphi(v/t)$

Pf. ($\Rightarrow$) given by specialization lemma

($\Leftarrow$) by contrapositive. Suppose $A \not\models \forall v\, \varphi(v)$. WTS: $A \not\models \varphi(v/t)$ for some t.

(*) - Then for some $x \in |A|$ we have $(A, x) \not\models \varphi(v/c_x)$

- By hypothesis there is a term t s.t. $t^A = x$.

- Since t written in language of A also have

$$t^{(A,x)} = x$$

i.e.

$$t^{(A,x)} = c_x^{(A,x)}$$ hence $(A,x) \models t \approx c_x$

By substitution lemma we get

$$(A,x) \models \varphi(v/c_x) \Leftrightarrow \varphi(v/t)$$

So by * we have

$$(A,x) \not\models \varphi(v/t)$$

and since $\varphi(v/t)$ written in language of A this gives:

$A \not\models \varphi(v/t)$ which is what we wanted to show.

<u>Lemma 3.5</u> Suppose t is a closed term and φ has only the free variable v.

Then $\varphi(v/t) \models \exists v \varphi$

<u>example</u>: Suppose t is $f(c)$ and φ is $c \leq v$ (where c constant symbol, f function symbol)

Then $\varphi(v/t)$ is

Lemma says: if A is a model in which $c < f(c)$, then $A \models \exists v\,(c \leq v)$.

<u>Pf of Lemma</u>: - Fix A and suppose $A \models \varphi(v/t)$.

- Suppose $t^A = x$
- then $(A, x) \models t \approx c_x$
- hence by substitution we have

$$(A, x) \models \varphi(v/t) \iff \varphi(v/c_x)$$

hence $(A, x) \models \varphi(v/c_x)$ by defn of $\models$ for $\iff$

hence $A \models \exists v \varphi(v)$ by defn of $\models$ for $\exists v$

Semantic Cemmer deduction + compactness completeness fun!

Generalization Lemma: - Spse Σ is a theory and $\varphi(v)$ is a fmla w/ a single free variable v

- Let c be a constant that does not occur in φ or in any sentence in Σ
- Suppose $\Sigma \models \varphi(v/c)$

Then $\Sigma \models \forall v \varphi$

i.e. if A is a struct in original language w/o c and $A \models \Sigma$ then $A \models \forall v \varphi$

i.e. if B is a structure in an expanded lang. that includes c and $B \models \Sigma$, then $B \models \varphi(v/c)$

Motivation: - this lemma corresponds to a common logical move in proofs

- often begin a proof: " Fix an arbitrary c (in our structure ...)

...

using axioms true in our structure ...

...

hence $\varphi(c)$

Since c was arbitrary (i.e. no matter how c is interpreted) we have $\forall v \varphi(v)$ "

<u>ex</u>: – Suppose $\Sigma = \{\forall u\, R(u, f(u)),$
$\forall u \forall v \forall w\, (R(u,v) \wedge R(v,w) \to R(u,w))\}$

– let φ be $R(v, f(f(v)))$
– Suppose $\mathcal{B}$ is a structure in a lang including R, f but also a constant c and $\mathcal{B} \models \Sigma$

– then $R^{\mathcal{B}}(c^{\mathcal{B}}, f^{\mathcal{B}}(c^{\mathcal{B}}))$ holds
– also $R^{\mathcal{B}}(f^{\mathcal{B}}(c^{\mathcal{B}}), f^{\mathcal{B}}(f^{\mathcal{B}}(c^{\mathcal{B}})))$
– hence $R^{\mathcal{B}}(c^{\mathcal{B}}, f^{\mathcal{B}}(f^{\mathcal{B}}(c^{\mathcal{B}})))$ holds
i.e. $\mathcal{B} \models R(c, f(f(c)))$
i.e. $\mathcal{B} \models \varphi(v/c)$
– Since $\mathcal{B}$ was arbitrary we have
$\Sigma \models \varphi(v/c)$
– Lemma says: $\Sigma \models \forall v\, \varphi(v)$

<u>pf of Lemma</u>: – Fix $\mathcal{A}$ (in same lang. as Σ w/o c), and suppose $\mathcal{A} \models \Sigma$.
– WTS $\mathcal{A} \models \forall v\, \varphi(v)$
i.e. for every $x \in |\mathcal{A}|$
$(\mathcal{A}, x) \models \varphi(v/c_x)$
– Fix $x \in |\mathcal{A}|$
– let $\mathcal{B}$ be an expansion of $(\mathcal{A}, x)$ with the constant c s.t. $c^{\mathcal{B}} = x$

has two new constants: c and c_x

- so we have $c^{\mathcal{B}} = x = c_x^{\mathcal{B}}$
- hence $\mathcal{B} \models c \approx c_x$ $(*)$

- Since $\mathcal{B}$ is an expansion of $\mathcal{A}$ we have $\mathcal{B} \models \Sigma$
- hence $\mathcal{B} \models \varphi(v/c)$ $(**)$
- but by substitution lemma and $(*)$ we have

$$\mathcal{B} \models \varphi(v/c) \iff \varphi(v/c_x)$$

- hence

$$\mathcal{B} \models \varphi(v/c_x) \quad \text{by } (**)$$

- hence

$$(\mathcal{A}, x) \models \varphi(v/c_x)$$

(since φ does not include c)

- hence, since $x \in |\mathcal{A}|$ was arbitrary, $\mathcal{A} \models \forall v \varphi(v)$ ✓

Lemma 3.7 - Let Σ be a theory and $\varphi(v)$ a fmla w/ a single free variable v.

- Suppose ψ is a sentence
- c a constant not appearing in ψ, φ, or anything in Σ

If: $\Sigma \cup \{\varphi(v/c)\} \models \psi$

Then: $\Sigma \cup \{\exists v \varphi(v)\} \models \psi$

Pf. later.

(i)

A Deduction System for FOL

- as in PL, deduction system we'll use is not canonical: there are other slightly different sets of rules we could use that would prove the same things
- just need some specific rules that allow us to "make the moves we want to make" in proofs for the symbols we use in FOL.

The rules

- Suppose Σ is a fixed theory
- All rules used in PL still apply (but now for first-order theories and sentences)
- need to add rules for equality $\approx$ and quantifiers.

These are:

$\approx$-In → really 3 separate deduction rules

$\{\} \vdash t \approx t$ (or write $\vdash t \approx t$)

$\{\} \vdash s \approx t \Leftrightarrow t \approx s$

$\{\} \vdash (r \approx s \wedge s \approx t) \Rightarrow r \approx t$

(ii)

$\approx$-out (substitution)

If s, t are closed terms and $\varphi(v)$ is a fmla w/ a single free variable v, then

$$\{s \approx t\} \vdash \varphi(v/s) \Leftrightarrow \varphi(v/t)$$

$\forall$-in (generalization)

If c is a constant that does not occur in φ or any sentence in Σ then

if $\Sigma \vdash \varphi(v/c)$ then $\Sigma \vdash \forall v\, \varphi$

$\forall$-out (specialization)

If t is a closed term and v is the only free variable in $\varphi(v)$, then

$$\{\forall v\, \varphi\} \vdash \varphi(v/t)$$

(**Duality**)

$\exists$-in: $\{\forall v \neg \varphi\} \vdash \neg \exists v\, \varphi$

$\exists$-out: $\{\neg \exists v\, \varphi\} \vdash \forall v \neg \varphi$

though at end of any deduction we could just collect all nonlogical symbols used and call that our language

Note: we do not require that our deductions stay within the language of our original theory

(iii)

Examples of deductions

- As before, a formal deduction is a sequence

$$\Sigma_0 \vdash \varphi_0$$
$$\Sigma_1 \vdash \varphi_1$$
$$\vdots$$
$$\Sigma_n \vdash \varphi_n$$

we'll start to check a bit.

where every line is justified by a single deduction rule (and possibly some previous lines in the deduction)

① Suppose r, s, t are closed terms. We show

$$\{r \approx s, s \approx t\} \vdash r \approx t$$

① $\{r \approx s, s \approx t\} \vdash r \approx s \wedge s \approx t$ $\quad \wedge$-in

② $\{\} \vdash (r \approx s \wedge s \approx t) \Rightarrow r \approx t$ $\quad \approx$-in

③ $\{r \approx s, s \approx t\} \vdash (r \approx s \wedge s \approx t) \Rightarrow r \approx t$ by R1 and ②

④ $\{r \approx s \wedge s \approx t, r \approx s \wedge s \approx t \Rightarrow r \approx t\} \vdash r \approx t$ $\quad \Rightarrow$-out

⑤ $\{r \approx s, s \approx t\} \vdash r \approx t$ by R2 ①, ③, ④ 2 Let φ be a sentence w/ a single free variable v. We show $\{\forall v \neg\neg\varphi\} \vdash \forall v \varphi$

PF. ① $\{\forall v \neg\neg\varphi\} \vdash \neg\neg\varphi(v/c)$ (specialization)
② $\{\neg\neg\varphi(v/c)\} \vdash \varphi(v/c)$ ($\neg$-elt) new constant symbol
③ $\{\forall v \neg\neg\varphi\} \vdash \varphi(v/c)$ (R2)
④ $\{\forall v \neg\neg\varphi\} \vdash \forall v \varphi$ (generalization)

Could also show: $\{\exists v \neg\neg\varphi\} \vdash \exists v \varphi$

3 We can also prove the "other" duality rules: ⓐ $\{\neg\forall v\varphi\} \vdash \exists v \neg\varphi$
ⓑ $\{\exists v \neg\varphi\} \vdash \neg\forall v\varphi$

Notice: no way of concluding a $\exists$ sentence directly, so we aim to deduce $\neg\neg\exists v \neg\varphi$

We do ⓐ:

① $\{\neg\forall v\varphi\} \cup \{\neg\exists v \neg\varphi\} \vdash \neg\forall v \varphi$ (R1)
② " " $\vdash \forall v \neg\neg\varphi$ ($\exists$-elt and R2)
③ " " $\vdash \forall v \varphi$ (by ② and R2)
④ $\{\neg\forall v \varphi\} \vdash \neg\neg\exists v \neg\varphi$ ($\neg$-in, lines ①, ③)
⑤ $\{\neg\forall v \varphi\} \vdash \exists v \neg\varphi$ ($\neg$-elt, R2)

We begin to combine steps here

(iv)

② (Example A from textbook)

- Suppose Σ is a theory, $\varphi(v)$ is a fmla w/ a single free variable v, ψ is a sentence
- Sps c is a constant not appearing in φ, ψ, or any sentence in Σ
- Sps $\Sigma \cup \{\varphi(v/c)\} \vdash \psi$
- We will show $\Sigma \cup \{\exists v \varphi\} \vdash \psi$

① $\Sigma \cup \{\varphi(v/c)\} \vdash \psi$ (by hypothesis)

② $\Sigma \vdash \varphi(v/c) \Rightarrow \psi$ $\Rightarrow$-in

check! ③ $\Sigma \vdash (\neg\psi) \Rightarrow (\neg\varphi(v/c))$ example 2.16 on HW R2

④ $\Sigma \cup \{\neg\psi\} \vdash$ by R1

⑤ $\Sigma \cup \{\neg\psi\} \vdash \neg\psi$ by R7

check! ⑥ $\Sigma \cup \{\neg\psi\} \vdash \neg\varphi(v/c)$ $\Rightarrow$-out, ④, ⑤, R2

⑦ $\Sigma \cup \{\neg\psi\} \vdash \forall v \neg\varphi$ generalization

check! ⑧ $\Sigma \cup \{\neg\psi\} \vdash \neg\exists v \varphi$ duality, R2

⑨ $\Sigma \vdash \neg\psi \Rightarrow \neg\exists v \varphi$ $\Rightarrow$-in

a lot of checking! ⑩ $\Sigma \vdash \exists v \varphi \Rightarrow \psi$ 2.16

⑪ $\Sigma \cup \{\exists v \varphi\} \Rightarrow \psi$ as in lines ④, ⑤, ⑥.

this corresponds to a semantic lemma 3.7 but with $\models$ replaced with $\vdash$

Soundness and Completeness

- in any deduction system, need soundness for it to be useful (don't want to be able to prove false sentences)
- Indeed for FOL we have soundness
- Also have finiteness of deductions
- we only sketch the proofs (similar to corresponding proofs in PL).

"obvious" since all our deduction rules intuitively preserve truth

Soundness Theorem: Fix a theory Σ and a sentence φ.
If $\Sigma \vdash \varphi$ then $\Sigma \models \varphi$.

Pf: - As in proof for PL, just want to check each deduction rule is sound,
- i.e. if we replace $\vdash$ with $\models$ in each rule we end up with a true statement.
- then in any formal deduction justifying $\Sigma \vdash \varphi$ can replace all $\vdash$'s with $\models$'s and get a real world proof that $\Sigma \models \varphi$.

- for all the old rules proof of soundness same as in PL: this is because we've defined Truth in the same way with respect to all the old symbols

- for new rules: we've already verified soundness for $\approx$-out (substitution) $\forall$-in (generalization), $\forall$-out (specialization) in prev. section

- remains to check soundness for $\approx$-in and duality rules

$\approx$-in is obvious: real $=$ is reflexive, symmetric, transitive

- the duality rules also clear. E.g. suppose we want to show the $\models$ rule for $\exists$-out, i.e.

$$[\neg \exists v \varphi] \models \forall v \neg \varphi$$

pf:

Sps $\mathcal{A} \models \neg \exists v \varphi$

then $\mathcal{A} \not\models \exists v \varphi$

i.e. there is no $x \in |\mathcal{A}|$ s.t. $(\mathcal{A}, x) \models \varphi(v/c_x)$

i.e. for every $x \in |\mathcal{A}|$ $(\mathcal{A}, x) \not\models \varphi(v/c_x)$

i.e. $(\mathcal{A}, x) \models \neg \varphi(v/c_x)$

hence $\mathcal{A} \models \forall v \neg \varphi$

$\to$ since $\mathcal{A}$ was arbitrary $[\neg \exists v \varphi] \models \forall v \neg \varphi$

$\boxtimes$

Some commentary:

- proving soundness may seem "circular" or "trivial," but conceptually something important going on:
  - Again: deduction rules + formal deductions are <u>purely</u> <u>syntactic</u> notions
  - That they have corresponding semantic forms is of course something we want (so things we prove are true in every structure satisfying axioms we reason from)
  - <u>but</u> deductions themselves do not depend on our def'ns of satisfaction or "truth in a model".
  - when we're ~~proving~~ performing a formal deduction we aren't handling real sets, functions, etc. — we're just pushing symbols

Theorem (Finiteness of deduction for FOL)
Suppose $\Sigma \vdash \varphi$. Then there is a finite subtheory $\Delta \subseteq \Sigma$ s.t. $\Delta \vdash \varphi$.

Pf: Deductions have finite length, every step uses only finitely many axioms from Σ.

(Really: induct on length of formal deduction justifying $\Sigma \vdash \varphi$)

Some more def'ns (same as in PL)

Suppose Σ is a theory:
- Σ is consistent iff $\perp$ is not deducible from $\perp$
- Σ is complete iff for every sentence φ written in lang. of Σ, either $\Sigma \vdash \varphi$ or $\Sigma \vdash \neg\varphi$.
- Σ is closed under deduction iff whenever $\Sigma \vdash \varphi$ then $\varphi \in \Sigma$.

Complete extensions Lemma

Suppose Σ is a consistent theory in a countable language.

There is a theory $\Gamma \supseteq \Sigma$ that is consistent, complete, and closed under deduction

for every φ either $\varphi \in \Sigma$ or $\neg\varphi \in \Sigma$.

Pf: (sketch: same as in PL)

- Enumerate all sentences in language: $\varphi_0, \varphi_1, \ldots$ (possible since lang is ctbl)
- Observe: for every φ, either $\Sigma \cup \{\varphi\}$ or $\Sigma \cup \{\neg\varphi\}$ is consistent

Lemma 2 φ only uses old prop'l rules

- Iteratively expand Σ by adding φ_n or $\neg\varphi_n$ at every stage so as to preserve consistency
- Let Γ be the theory we end up with: contains φ or $\neg\varphi$ for every φ, and is consistent (o.w. by finite deductions some finite subtheory inconsistent - impossible by our construction)

(i)

Another useful tool that allows us to exploit completeness theorem of PL in first-order deduction

<u>Tautology</u> <u>lemma</u> If φ is a tautology then $[] \vdash \varphi$.

<u>Pf.</u> • φ is of the form $\pi(P_0/\psi_0, \ldots P_n/\psi_n)$, where π is a valid prop'l sentence and $\psi_0, \ldots, \psi_n$ are the first-order sentences replacing the variables in π

takes place in PL

- Since π valid we have $[] \models \pi$
- hence by completeness $[] \vdash \pi$
- but any formal deduction of π yields a formal deduction of $\pi(P_0/\psi_0, \ldots, P_n/\psi_n)$ — i.e. of φ. ✓
- hence $[] \vdash \varphi$.

<u>ex:</u> - $\neg\neg R(c) \Rightarrow R(c)$ is a tautology
- underlying PL sentence is $\neg\neg P \Rightarrow P$

(ii)

- by completeness we know there exists a deduction of $\neg\neg P \Rightarrow P$, here's a specific one:

$$\{\neg\neg P\} \vdash P \qquad \neg\text{-out}$$
$$\{\} \vdash \neg\neg P \Rightarrow P \qquad \Rightarrow\text{-in}$$

- but if we replace every instance of P with $R(c)$ in above we get a deduction of
$$\{\} \vdash \neg\neg R(c) \Rightarrow R(c)$$

Tautology lemma says this can always be done.

Completeness of FOL

- unlike soundness + related theorems completeness of FOL doesn't boil down to straightforward adaptation of PL version

- reason: totally new idea of structure and truth in a structure

- Still: strategy is similar main goal is to prove consistent theories have models (model existence)

(iii)

- need a few preliminary lemmas first.

<u>Lemma 3.12</u> Suppose Σ is consistent, $\varphi(u)$ is a fmla w/ a single free variable u, c is a constant symbol not occurring in φ or Σ.

Let η be $(\exists u \varphi) \Rightarrow \varphi(u/c)$

Then $\Sigma \cup \{\eta\}$ is consistent

<u>Pf</u>: - Sps not
- then $\Sigma \cup \{\eta\} \vdash \psi$
$\Sigma \cup \{\eta\} \vdash \neg\psi$ for any ψ
- hence $\Sigma \vdash \neg\eta$

It follows that
$\Sigma \vdash \exists u \varphi$ and $\Sigma \vdash \neg\varphi(u/c)$

(This is tautology lemma in action:
$\Sigma \vdash \neg(\exists u \varphi \Rightarrow \varphi(u/c))$
$\Rightarrow (\exists u \varphi \wedge \neg\varphi(u/c))$
since this is a tautology
hence $\Sigma \vdash \exists u \varphi \wedge \neg\varphi(u/c)$ by $\Rightarrow$-out)

hence $\Sigma \vdash \forall u \neg\varphi$ $\forall$-in
hence $\Sigma \vdash \neg\exists u \varphi$ duality
a contradiction, as $\Sigma \vdash \exists u \varphi$ and Σ is consistent ✓

(i)

- Next goal: prove if Σ is consistent then there is A s.t. $A \models \Sigma$ (model existence)

- main step in proving completeness.

- Remember how we proved model existence in PL:
  - given consistent Σ, extend to complete, consistent Σ^* that actually contains φ or $\neg\varphi$ for every φ
  - then Σ^* contains P_i or $\neg P_i$ for every $i \in \omega$.
  - Define $A(P_i) = 1$ iff $P_i \in \Sigma$.
  - This is only possible model, but it works: $A \models \Sigma^*$ hence $A \models \Sigma$.

- Beautifully, similar idea works in FOL, but much harder to implement.
  - Given consistent Σ, we will extend to a complete, consistent Σ^* called a Henkin extension
  - Key point: Σ^* is written in an expanded language where for every existential statement $\exists v \varphi$ in Σ^* we have a new constant symbol c_φ.
  - The symbol c_φ will itself be the witness of $\exists v \varphi$ in the model A we define

(ii)

idea: let universe be set of constants then look to theory to say when they're related.

i.e.

- $|A|$ = set of equivalence class of constants in Σ^*
- $c^A = c$ for all c
- R^A determined by Σ^* for all R
- f^A determined by Σ^* for all f

We'll prove that this A works, i.e. $A \models \Sigma^*$, hence $A \models \Sigma$.

Ex. 1 - Let R be a unary relation symbol and c a constant symbol.

- Let $\Sigma_0 = \{R(c)\}$ (Σ_0 is consistent)
- What is a model for Σ_0?

Define $A = \langle |A|, c^A, R^A \rangle$ where

- $|A| = \{c\}$
- $c^A = c$
- R^A is relation on $|A|$ that is true at c^A, i.e. $R^A = \{c\}$

Then $A \models \Sigma_0$! (why: Clearly $A \models R(c)$ since $R^A(c^A)$ is true)

Ex. 2: Can't always get away with letting $|A|$ = set of constants appearing in theory.

(iii)

- Let $\Sigma_1 = \{R(c), \neg(c \approx b), c \approx d\}$
- Can't have our model be set of constant symbols interpreted as themselves (c, d are distinct symbols but need $c^A = d^A$)

- Instead introduce equiv. relation on $\{b, c, d\}$ as suggested by Σ_1:
  $b \not\sim c$, $b \not\sim d$
  $c \sim d$.

- Then $[c] = [d] = \{c, d\}$
  $[b] = \{b\}$

- Define $A = \langle |A|, c^A, d^A, b^A, R^A \rangle$
  - $|A| = \{[b], [c]\}$
  - $c^A = d^A = [c]$ $\quad b^A = [b]$
  - R^A is true at c^A and d^A but not b^A
    i.e. $R^A = \{[c]\}$

Then $A \models \Sigma_1$

↳ notice: - could have defined $R^A = \{[b], [c]\}$
- would still have $A \models \Sigma_1$
- in practice we will expand Σ_1 to "make this decision for us"
- e.g. define $\Sigma_1^* = \Sigma_1 \cup \{R(b)\}$
- then only our second ex works

(iv)

ex 3: – Let $\Sigma_2 = \{R(c), R(b), c \approx d, \neg(c \approx b), \exists u \neg R(u)\}$

– Can't have a model A with $|A| = \{[b], [c]\}$ because we will have no witness to $\exists u \neg R(u)$.

– So we introduce a witness!

– Let φ denote $\neg R(u)$
– Let c_φ be a new constant symbol, and let $[c_u] = [c_\varphi]$.

– Define $A = \langle |A|, c^A, d^A, b^A, c_\varphi^A, R^A \rangle$ with
$c^A = d^A = [c]$
$b^A = [b]$
$c_\varphi^A = [c_\varphi]$

– $R^A = \{[c], [b]\} = [c^A, b^A)$

– Then $A \models \Sigma_2$.

main point being
$A \models \exists u \neg R(u)$
because $c_\varphi^A = [c_u]$ is the witness!

– Can adapt strategy of ex.3 to prove model existence in general

– Idea is: given consistent Σ extend to complete Σ^* in a lang.

that includes a constant symbol c_φ for every existential claim $\exists v \varphi$
– a bit tricky: for "deductive reasons" we first expand to include all sentences of the form $\exists v \varphi \Rightarrow \varphi(^v/c_\varphi)$

Henkin Extension lemma: Let Σ be a consistent theory in a countable language. (Hence Σ is ctbl.)
There is a consistent extension Σ^H of Σ, in an expanded lang. with new constant symbols, s.t. for every fmla $\varphi(v)$ in the expanded language with a single free variable v, there is a constant symbol c_φ s.t. the sentence

new lang. still ctbl.

$$\exists v \varphi \Rightarrow \varphi(^v/c_\varphi)$$

is in Σ^H.

Pf: – Prev. lemma: if we add such a sentence to a consistent theory we maintain consistency
– so let's just recursively add all such sentences.

Define a chain of theories:

$$\Sigma = \Sigma_0 \subseteq \Sigma_1 \subseteq \cdots$$

(vi)

as follows: If Σ_n is defined, then for every fmla $\varphi(v)$ written in lang of Σ_n with a single free variable v let c_φ be a new constant symbol and let

η_φ be $\exists v \varphi \Rightarrow \varphi(v/c_\varphi)$

Let $\Sigma_{n+1} = \Sigma_n \cup \{\eta_\varphi \mid \varphi \text{ as above}\}$

- Note: if lang. of Σ_n is ctbl, then set of η_φ is ctbl, hence if Σ_n is ctbl too, Σ_{n+1} is ctbl.
- And Σ_0 is ctbl in a ctbl. lang. hence all Σ_n's ctbl.
- Further, if Σ_n is consistent, then Σ_{n+1} is too:
  - Σ_{n+1} obtained by adding ctbly many sentences each of which is consistent w/ Σ_n and all prev. sentences so by finiteness of deduction Σ_{n+1} consistent.

Now: let $\Sigma^H = \bigcup_{n \in \omega} \Sigma_n$

- Then Σ^H is ctbl and by finiteness of deduction also consistent, since all Σ_n are.

- Remains to show Σ^H has the

(vii)

- Remains to show Σ^H has the desired "Henkin Property"
- Let $\varphi(v)$ be a fmla w/ a single free variable written in lang. of Σ^H.
- Then there is n s.t. $\varphi(v)$ is written in lang. of Σ_n.
- hence $\exists v\, \varphi \Rightarrow \varphi(v/c_\varphi)$ is in Σ_{n+1} and therefore also in Σ^H ✓

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Theorem (Model Existence for FOL)
If Σ is a consistent theory, then Σ has a model.

Throw in all constants then extend to Σ^* to "make decisions for us"

Pf: – **First**: Use Henkin Extension Lemma to extend Σ to Σ^H (in expanded lang.)
– **Then**: Since Σ^H is (still) consistent, can extend again to a complete, consistent theory Σ^* that is closed under deduction.

For every sentence φ, exactly one of $\varphi \in \Sigma^*$ or $\neg\varphi \in \Sigma^*$ holds

– **Note**: in going $\Sigma^H \to \Sigma^*$ we don't further expand language.
– Hence still have: if $\varphi(v)$ is fmla w/ a single free variable written in expanded lang. then $\exists v \varphi \Rightarrow \varphi(v/c_\varphi) \in \Sigma^*$.

– **Idea** is: Σ^* strong enough to "determine" a model A.
– Our underlying universe won't just be set of constants but actually (equiv. classes of) closed *terms*

(doing terms instead of constants just for notational convenience)

- Let S = set of closed terms in language of Σ^*.
- for $s, t \in S$ define $s \sim t$ iff the sentence $s \approx t \in \Sigma^*$.

Claim $\sim$ is an equiv. relation on S

Pf: - fix $r, s, t \in S$
- Since Σ^* is cons., comp., closed under $\vdash$

a) $r \approx r \in \Sigma^*$ (hence $r \sim r$)

b) if $r \sim s$ then $r \approx s \in \Sigma^*$ hence $s \approx r \in \Sigma^*$ (by $\approx$-in)
hence $s \sim r$

c) if $r \sim s, s \sim t$ then
$r \approx s, s \approx t \in \Sigma^*$ hence (by $\approx$-in)
$r \approx t \in \Sigma^*$ hence
$r \sim t$

- We define a structure A in our expanded lang.

- $|A| = \{[s] : s \in S\}$
- if c is a constant symbol, we let $c^A = [c]$
- if F is an n-ary function symbol and $s_1, .., s_n, t$ closed terms we let
$F^A([s_1], .., [s_n]) = [t]$ iff
$F(s_1, .., s_n) \approx t \in \Sigma^*$
- if R is an n-ary relation symbol, $s_1, .., s_n$ closed terms we define $R^A([s_1], .., [s_n])$ iff
$R(s_1, .., s_n) \in \Sigma^*$

Claim (A): This defines a structure A in the lang. of Σ^*.

(i)

- need to check that these def/ns really work, i.e. don't depend on representative of equiv. class

- For simplicity: assume F is a unary function symbol, s, s', t, t' closed terms s.t. $s \sim s'$, $t \sim t'$ (hence $s \approx s'$, $t \approx t' \in \Sigma^*$)

<u>Mini-claim</u>: $F(s) \approx t \in \Sigma^*$ iff $F(s') \approx t' \in \Sigma^*$

<u>Pf</u>: ($\Rightarrow$) - sps $F(s) \approx t \in \Sigma^*$

- since $t \approx t' \in \Sigma^*$, $\Sigma^* \vdash F(s) \approx t'$ ← $\approx$-in

closed under deduction! hence $F(s) \approx t' \in \Sigma^*$

- since $s \approx s' \in \Sigma^*$ by $\approx$-out (subs) $\Sigma^* \vdash F(s) \approx t' \Leftrightarrow F(s') \approx t'$

- hence $\Sigma^* \vdash F(s') \approx t'$

hence $F(s') \approx t' \in \Sigma^*$.

($\Leftarrow$) completely symmetric

More generally: if F, R are n-ary, we have terms $s_i \sim s_i'$ and $t \sim t'$ then

$F(s_1, \ldots, s_n) \approx t \in \Sigma^*$ iff
$F(s_1', \ldots, s_n') \approx t' \in \Sigma^*$

and $R(s_1, \ldots, s_n) \in \Sigma^*$ iff
$R(s_1', \ldots, s_n') \in \Sigma^*$.

Hence our def/ns of F^A and R^A are legitimate and our structure A is well-defined in expanded lang.

Claim (B): for every closed term t, we have $t^{A} = [t]$

Pf: – easy induction on complexity of t (leave it to you)
– base case is when t is a constant c, then by def'n $c^{A} = [c]$.

Claim (C): For every sentence φ (in expanded lang.) we have
$A \models \varphi$ iff $\varphi \in \Sigma^*$

Pf.: – induction on complexity of φ
– may assume φ is written using only $\exists, \neg, \wedge$
– Why: every sentence ψ is provably and logically equivalent to such a φ
(we still assume other sentences are there, but only prove (C) for sentences w/ $\neg$, $\wedge$, $\exists$)

Base case ① φ is $R(s_1, .., s_n)$
Then $A \models R(s_1, .., s_n)$ iff
$R^{A}(s_1^{A}, .., s_n^{A})$ iff
$R^{A}([s_1], .., [s_n])$ iff (by def'n)
$R(s_1, .., s_n) \in \Sigma^*$ i.e. iff
$\varphi \in \Sigma^*$

② φ is $s \approx t$
Then $A \models s \approx t$ iff
$s^{A} = t^{A}$ iff (by Claim (B))
$[s] = [t]$ iff
$s \sim t$ iff $s \approx t \in \Sigma^*$ i.e. $\varphi \in \Sigma^*$

Sps claim is true for φ, ψ

(1) Consider $\neg\varphi$.
$A \models \neg\varphi$ iff
$A \not\models \varphi$ iff (by induction)
$\varphi \notin \Sigma^*$ iff (by completeness/closure under deduction)
$\neg\varphi \in \Sigma^*$ ✓

(2) Consider $\varphi \wedge \psi$
$A \models \varphi \wedge \psi$ iff
$A \models \varphi$ and $A \models \psi$ iff (by induction)
$\varphi \in \Sigma^*$ and $\psi \in \Sigma^*$ iff (by $\wedge$-in $\wedge$-out)
$\Sigma^* \vdash \varphi \wedge \psi$ iff
$\varphi \wedge \psi \in \Sigma^*$ ✓

- Now suppose our sentence is $\exists v \varphi$
- In this case our IH is that claim holds for all sentences of the form $\varphi(v/c)$

WTS: $\exists v \varphi \in \Sigma^*$ iff $A \models \exists v \varphi$

($\Rightarrow$) - Sps $\exists v \varphi \in \Sigma^*$
- we know there is a constant c_φ and sentence $\exists v \varphi \Rightarrow \varphi(v/c_\varphi) \in \Sigma^*$
- hence $\Sigma^* \vdash \varphi(v/c_\varphi)$ by modus ponens
- hence $\varphi(v/c_\varphi) \in \Sigma^*$
- by IH $A \models \varphi(v/c_\varphi)$
- by lemma 3.5 $A \models \exists v \varphi$ ✓

($\Leftarrow$) - sps $\exists v \varphi \notin \Sigma^*$

- then $\neg \exists v \varphi \in \Sigma^*$ (completeness, closure under $\vdash$)

- hence $\forall v \neg \varphi \in \Sigma^*$ (duality, closure under $\vdash$)

- hence for every closed term t, $\neg \varphi(v/t) \in \Sigma^*$ (specification, $\Rightarrow$-out, closure under $\vdash$)

- by IH, for every closed term t

$A \models \neg \varphi(v/t)$

- by lemma 3.4 $A \models \forall v \neg \varphi$

hence $A \not\models \exists v \varphi$ ✓ ($\nLeftarrow$), cshfs

This completes the induction

Hence $A \models \Sigma^*$

Hence, if A' is reduction of A to original language

$A' \models \Sigma$ ✓

- May not be entirely clear from proof how we're dealing w/ universal sentences

- <u>Quick</u> <u>example</u>:
  - Sps R is a unary relation symbol and $\forall u R(u) \in \Sigma^*$.
  - Then $\Sigma^* \vdash R(t)$ for every closed term t by specialization hence $R(t) \in \Sigma^*$ for all closed t
  - hence $R^A([t])$ is true by <u>defn</u> of R^A
  - hence for all t, $A \models R(t)$ hence $A \models \forall u R(u)$ by 3.4. ✓

We prove completeness and compactness from model existence exactly as in PL.

<u>Completeness theorem for FOL</u>,

If $\Sigma \models \varphi$ then $\Sigma \vdash \varphi$.

<u>Pf</u>: - Sps ~~[illegible]~~ $\Sigma \nvdash \varphi$
- then $\Sigma \cup \{\neg\varphi\}$ is consistent.
- let $A \models \Sigma \cup \{\neg\varphi\}$ then $A \models \Sigma$ and $A \not\models \varphi$ hence $\Sigma \not\models \varphi$.

Compactness theorem for FOL
If every finite subtheory of Σ has a model, then so does Σ.

Pf: Suppose Σ has no model
Then Σ is inconsistent, by ~~model~~ model existence
Hence some finite subtheory Δ is inconsistent, by finiteness of deductions
Hence Δ has no model, by soundness.

Example (3.6) Suppose Σ is a theory with arbitrarily large finite models. Then Σ has an infinite model.

Pf. - For every $n \geq 1$ let φ_n be the sentence

$$\exists v_1 \ldots \exists v_n \left(\bigwedge_{i \neq j} v_i \neq v_j \right)$$

- φ_n asserts there are (at least) n distinct elements in the universe

- Let $\Pi = \{\varphi_n : n \geq 1\}$

- We prove $\Sigma \cup \Gamma$ is finitely satisfiable

- Fix $\Delta \subseteq \Sigma \cup \Gamma$ finite

$$\Delta = \Delta_0 \cup \Delta_1$$
$$\Delta_0 \subseteq \Sigma \quad \Delta_1 \subseteq \Gamma$$

- there is N s.t. $\forall m \geq N$ $\varphi_m \notin \Delta_1$

- let A be a model of Σ of size $\geq N$

- Then $A \models \Delta_0 \subseteq \Sigma$ and $A \models \Delta_1$ since

- hence $A \models \Delta$ ✓

- By compactness $\Sigma \cup \Gamma$ has a model B

- any such model B is infinite since $B \models \Gamma$ <u>and</u> a model of Σ ✓