First-order Logic (FOL)

- substantially more expressive logical system than PL.

- sentences in FOL have same "logical structure" as PL

$$((\cdots) \vee (\cdots)) \Rightarrow (\cdots)$$

- except now the $(\cdots)$ won't be prop variables but basic statements involving constants, functions, and relations

- and we sprinkle in quantifiers

$$\forall x\ ((x \geq 1) \vee (x \leq -1) \Rightarrow x^2 \geq 1)$$

- main objects of study will be first-order structures these are sets with distinguished constants, functs, relations

e.g. $(\mathbb{R}, 0, 1, \pi, +, \exp, \leq)$

Syntax of FOL

logical symbols

$\bot \ \top \ \neg \ \wedge \ \vee \ \Rightarrow \ \Leftrightarrow \ \forall \ \exists$

variables

$v_0, v_1, \ldots$ (lowercase now)

↳ technically all variable symbols come from above list, but we sometimes write $x, y, z, u, v, \ldots$ for variables

equality symbol

$\approx$ (will usually write $=$)
(point is to distinguish from "actual" equality)

in addition to above symbols (can always use) we will work always w.r.t. a language L
languages consist of constant symbols ($c, d, \ldots$), function symbols ($f, g, \ldots$) and relation symbols ($R, S, \ldots$)

- each function and relation symbol has a fixed arity $n \geq 1$

Sometimes will write
$L = \langle c_0, c_1, \ldots, f_0, f_1, \ldots, R_0, R_1, \ldots \rangle$

e.g. $\langle 0, 1, +, \exp, \leq \rangle$ is a language with two constants (0, 1)
a binary function symbol +
a unary function symbol exp
a binary relation symbol $\leq$.

just symbols for now: no meaning!!

To do: define **terms** $\rightsquigarrow$ objects in our structure
formulas $\rightsquigarrow$ assertions that are T/F depending on specification of variables
sentences $\rightsquigarrow$ ~~sssth~~ assertions always T/F

objects → terms
statements → formulas

- Terms intuitively stand for objects in a structure
- defined recursively
  - every variable v_i is a term
  - every constant is a term
  - if f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms then $f(t_1, \ldots, t_n)$ is a term

e.g

- v_0
- 1
- $\exp(v_0+1)$ are terms in $\langle 0, 1, +, \exp, \leq \rangle$

should write $+(v_0, 1)$

- Formulas also defined recursively

atomic formulas

- $\top$ and $\bot$ are formulas
- if s, t are terms, then $s \approx t$ is a formula
- if R is an n-ary relation symbol and $t_1, \dots, t_n$ are terms then $R(t_1, \dots, t_n)$ is a term

- if φ and ψ are formulas, so are

  $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \Rightarrow \psi$, $\varphi \Leftrightarrow \psi$

  $\neg \varphi$

  $\forall v_i \, \varphi$

  $\exists v_i \, \varphi$

e.g. if our language is $(0, 1, +, \exp, \leq)$
then

$$1 + v_5 \quad , \quad \exp(v_7) \quad , \quad v_7 , 0$$

are terms

$$1 + v_5 \leq \exp(v_7) \quad , \quad v_7 = 0$$

~~then~~ are atomic formulas

$$\forall v_5 \left(1 + v_5 \leq \exp(v_7) \Rightarrow v_7 = 0 \right)$$

is a formula

→ sometimes use x, y, z, u, v for variables for readability

e.g. $\forall x \left(1 + x \leq \exp(y) \Rightarrow y = 0 \right)$

Free variables

If φ is $u+u \leq 1$ (should write: $\leq(+(u,u),1)$), we say u is free in φ since intuitively truth of φ depends on specification of u.

OTOH u is not free in $\forall u(u+u \leq 1)$ since intuitively this formula is outright true/false.

Inductively we define this idea of "occurs freely in" (basically: u occurs freely until it's quantified out)

- no variables occur freely in $\top$ or $\bot$
- u occurs freely in $s=t$ iff u occurs in s or t
- u occurs freely in $R(t_0, \ldots, t_{n-1})$ iff occurs in t_i for some $i<n$
- u occurs freely in $\neg\varphi$ iff u occurs freely in φ.
- u occurs freely in $\varphi * \psi$ iff u occurs freely in φ or ψ, where $*$ is $\wedge$ $\vee$ $\rightarrow$ or $\leftrightarrow$.
- u occurs freely in $\forall v\varphi$ and $\exists v\varphi$ iff u occurs freely in φ and $u \neq v$

the point: if u is free in φ then u is no longer free in $\forall u \varphi$ and $\exists u \varphi$.

e.g. u occurs freely in $u+v=7$ but u is not free in $\forall u(u+v=7)$ but v is still free in this formula.

Warning: u does occur freely in

$$\left[\forall u(u+u \geq u)\right] \wedge u=3$$

bound here   free here.

This is a well-formed formula, but would be bad actual mathematical style to reuse a single variable in this way.

- a sentence is a formula w/ no free variables

e.g. $0=1$ and $\forall u(u=u)$ are sentences but $u=1$ is not.

Substitution

Often when calculating, useful to substitute:

e.g. $u = 7^2$

$v + u = 12$

$v + 7^2 = 12$

Before we can do "proofs" like this, need to define syntactically what it means to substitute terms for variables

e.g. if t is the term $v+u$ ~~[illegible]~~ and r is the term 7^2, we'll write $t(u/r)$ for the term $v + 7^2$.

Substitution into terms:

Suppose t is a term and u is a variable and r is a term:

- if u does not occur in t then $t(u/r)$ is t
- if t is the variable u then $t(u/r)$ is r.
- if t is $F(t_0, \ldots, t_{n-1})$ then $t(u/r)$ is $F(t_0(u/r), \ldots, t_{n-1}(u/r))$

e.g. if t is $\exp(u+v)$ (iv)
then $t(u/1+x)$ is $\exp((1+x)+v)$

Substitution into formulas.

Same idea: e.g. if φ is the fmla $u+7=v$
and r is the term $x+1$ then
$\varphi(u/r)$ is $(x+1)+7=v$

inductive defi'n.

Sps φ a fmla, u a variable, r a term

- if φ is $\bot$ or $\top$ then $\varphi(u/r)$ is φ
- if φ is $s=t$ then $\varphi(u/r)$ is $s(u/r)=t(u/r)$
- if φ is $R(t_0,\dots,t_{n-1})$ then $\varphi(u/r)$ is $R(t_0(u/r),\dots,t_{n-1}(u/r))$
- if φ is $\neg\psi$ then $\varphi(u/r)$ is $\neg\psi(u/r)$
- if φ is $\psi * \chi$ then $\varphi(u/r)$ is $\psi(u/r) * \chi(u/r)$ where $*$ is $\wedge\ \vee\ \Rightarrow$ or $\Leftrightarrow$
- if φ is $\exists v\,\psi$ then:
  - if u is v then $\varphi(u/r)$ is φ
  - if u is not v then $\varphi(u/r)$ is $\exists v\,\psi(u/r)$
- if φ is $\forall v\,\chi$
  - if u is v then $\varphi(u/r)$ is φ
  - if u is not v then $\varphi(u/r)$ is $\forall v\,\chi(u/r)$

point of last two bullets: non-free occurrence of a ~~variable~~ variable can't be substituted!

<u>More examples:</u>

Suppose φ is the formula $R(u,v)$ where R is a binary relation symbol and u,v are distinct variables, c,d are distinct constants

- $\varphi(u/v)$ is $R(v,v)$
- $\varphi(v/u)$ is $R(u,u)$
- $\varphi(u/v)(v/u)$ is also $R(u,u)$
- $\varphi(v/u)(u/v)$ is $R(v,v)$
- $\varphi(u/c)$ is $R(c,v)$
- $\varphi(v/d)$ is $R(u,d)$
- $\varphi(u/c)(v/d)$ is $R(c,d)$
- $\varphi(v/d)(u/c)$ is $R(c,d)$ also

Now suppose φ is
$$(\forall u R(u,v)) \vee (\exists v S(u,v))$$
where S is another binary relation symbol.

Then $\varphi(u/v)$ is $(\forall u R(u,v)) \vee \boxed{\exists} v S(v,v)$
$\varphi(v/u)$ is $(\forall u R(u,u)) \vee (\exists v (S(u,v))$

(i)

Structures

- Given a language, e.g. $\langle 0, 1, +, \exp, \leq \rangle$, ~~[illegible]~~ a structure in this language is a set in which we interpret these symbols as actual constants, functions, and relations

- Before interpretation, symbols are just symbols

- the same symbol may be interpreted in different ways over two different structures

Formal defn.

- Fix a language L

- a structure A in this language consists of:
    - a set $|A|$ (called the universe of A)
    - for each $c \in L$, an element $c^A \in |A|$

(ii)

- For each n-ary function symbol $F \in L$, an n-ary function
  $$F^A : |A|^n \to |A|$$
- For each n-ary relation symbol $R \in L$, an n-ary relation $R^A \subseteq |A|^n$.

Examples

① Let $L = \langle 0, 1, +, \exp, \leq \rangle$

Consider the structure A in which $|A| = \mathbb{R}$,

0^A is "the real 0"
1^A is "the real 1"
$+^A$ is " $+$ "
$\exp^A$ is " exponential function"
$\leq^A$ is " less than "

then $A = \langle \mathbb{R}, 0^A, 1^A, +^A, \exp^A, \leq^A \rangle$ is a structure in this language.

② Now consider the structure B in which $|B| = \mathbb{R}$, and $0, 1, \exp, \leq$ are interpreted as before, but $+^B$ is the function defined by $x +^B y = x \cdot y - 2$.

(iii)

Then $B = \langle \mathbb{R}, 0^B, 1^B, +^B, \exp^B, \le^B \rangle$
is also a structure in this language.

We have
$$(5 +^A 7) +^A 2 = 14$$
but
$$(5 +^B 7) +^B 2 = (5 \cdot 7 - 2) \cdot 2 - 2$$
$$= 64$$

(3) - Let $L = \langle c, R \rangle$ be a language with a single constant symbol and single binary relation symbol.

- Let T denote the set of finite 01-functions:
$$T = \{ f \mid \mathrm{dom}(f) = n \text{ for some } n \in \omega,\ \mathrm{ran}(f) \subseteq \{0,1\} \}$$

includes $\emptyset$

think of el'ts as sequences
e.g. $\langle 0, 0, 1 \rangle$
$\langle 0, 0, 1, 0 \rangle$.

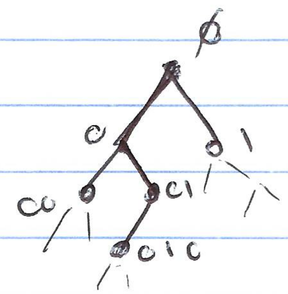
- Let A be the structure with $|A| = T$
$$c^A = \emptyset$$

(iv)

and R^A is the "extension" relation

i.e. $R^A(f,g)$ iff $dom(f) \subseteq dom(g)$
and $g \restriction dom(f) = f$

e.g. $R^A(\langle 0,1\rangle, \langle 0,1,0\rangle)$ is true
but not $R^A(\langle 0,0\rangle, \langle 0,1,0\rangle)$



Then $A = \langle T, c^A, R^A\rangle$
is the "rooted binary tree ordered by end extension."

<u>a nonexample</u>

- functions for us in a structure A go from $|A|^n$ (for some n) <u>to</u> <u>$|A|$</u>

- e.g. suppose $L = \langle f\rangle$ is a language with a single function symbol.

(v)

- then if A is the "structure" with $|A| = \mathbb{R}$ and $f^A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f^A(x,y) = (x+y, x-y)$ then $A = \langle \mathbb{R}, f^A \rangle$ is not a structure at all.

(not an important issue, just pointing it out)

Semantics:

- Now want to define $\text{Truth}_A(\varphi)$ (sentence) for a structure A
  - ↳ roughly: $\text{Truth}_A(\varphi) = 1$ iff φ is true (relative to A's interp. of symbols in φ).

- First need to say how to eval. closed terms (recall: a term is closed if it contains no variables)

  ↳ if t is a closed term, t^A is defined inductively:
  - if t is c a constant then t^A is c^A
  - if t is $f(t_1, \ldots, t_n)$ then t^A is $f^A(t_1^A, \ldots, t_n^A)$

(vi)

usual interps

e.g. if $A = \langle \mathbb{R}, 1^A, +^A \rangle$
and t is the term
$(1+1)+1$

then
$$t^A = (1+1)^A +^A 1^A$$
$$= (1^A +^A 1^A) +^A 1^A$$
$$= 3$$

<u>Expanding by a new constant:</u>

- need the following to deal w/ sentences w/ quantifiers

- Sps A is a structure and $x \in |A|$
- can't refer to x explicitly in formulas unless $c^A = x$ for some constant symbol c.
- if there is no such c, can introduce a new symbol c_x and define $c_x^A = x$.

- technically we have defined a new structure, which we call (A, x)

(vii)

So that:

- $c^{(A,x)} = c^A$ for all $c \neq c_x$
- $f^{(A,x)} = f^A$
- $R^{(A,x)} = R^A$
- $c_x^{(A,x)} = x$

$\boxed{\text{* See example next page}}$

- With this notation can finally define Truth_A

- $\mathrm{Truth}_A : \{\varphi : \varphi \text{ a sentence}\} \to \{0,1\}$ is unique function s.t.
  - $\mathrm{Truth}_A(\top)=1 \quad \mathrm{Truth}_A(\bot)=0$
  - if φ is $s \approx t$ then $\mathrm{Truth}_A(\varphi)=1$ iff $s^A = t^A$

    must be closed terms if φ is a sentence

  - if φ is $R(t_1,\dots,t_n)$ then $\mathrm{Truth}_A(\varphi)=1$ iff $R^A(t_1^A,\dots,t_n^A)$

  - Truth_A is defined as in PL for $\neg\varphi \quad \varphi\wedge\psi \quad \varphi\vee\psi \quad \varphi\Rightarrow\psi \quad \varphi\Leftrightarrow\psi$

  - If φ is $\exists u\,\psi$

<u>example</u>: Suppose $L = \langle 0, 1, +, \times \rangle$ and $A = \langle \mathbb{R}, 0^A, 1^A, +^A, \times^A \rangle$ where all interpretations are the usual ones.

We can't write
"$\sqrt{2} \times \sqrt{2} = 2$"
in this language.

So we expand our structure.

$(A, \sqrt{2})$ is a structure in the language $\langle 0, 1, c_{\sqrt{2}}, +, \times \rangle$

We have
$$(A, \sqrt{2}) = \langle 0^{(A,\sqrt{2})}, 1^{(A,\sqrt{2})}, c_{\sqrt{2}}^{(A,\sqrt{2})}, +^{(A,\sqrt{2})}, \times^{(A,\sqrt{2})} \rangle$$

where all interps are same as A but now $c_{\sqrt{2}}^{(A,\sqrt{2})} = \sqrt{2}$

We have
$$(A, \sqrt{2}) \models c_{\sqrt{2}} \times c_{\sqrt{2}} = 1 + 1$$

In order to write 2 directly, would need another symbol...

(viii)

then $\text{Truth}_A(\varphi) = 1$ iff there exists $x \in |A|$ s.t. $\text{Truth}_{(A,x)}(\psi(u/c_x)) = 1$

• if φ is $\forall u \psi$ then $\text{Truth}_A(\varphi) = 1$ iff for every $x \in |A|$, $\text{Truth}_{(A,x)}(\psi(u/c_x)) = 1$

<u>example</u> let $A = \langle \mathbb{R}, 0^A, 1^A, +^A \rangle$

① if φ is $1 + 1 = 0$

then $\text{Truth}_A(\varphi) = 0$

since $1^A +^A 1^A \neq 0^A$

(but $\text{Truth}_A(\neg\varphi) = 1$)

② if φ is $\exists u (u + 1 = 0)$

then $\text{Truth}_A(\varphi) = 1$

<u>why</u>: let ψ be $u + 1 = 0$

Introduce a new symbol c_{-1} and interpret c_{-1} as -1 in the expanded structure (A, c_{-1})

Then

~~then~~ $\psi(u/c_{-1})$ is

$c_{-1} + 1 = 0$

Hence $\text{Truth}_{(A,c_{-1})}(\psi(u/c_{-1})) = 1$

(ix)

Hence $\text{Truth}_A(\varphi) = 1$ ✓

Some more terminology (same as PL)

- a set of sentences Σ is called a theory
- $A \models \varphi$ means $\text{Truth}_A(\varphi) = 1$

  how we usually indicate $\text{Truth}_A(\varphi) = 1$
- $A \models \Sigma$ iff $A \models \varphi$ for all $\varphi \in \Sigma$
- $\varphi \models \psi$

  $\varphi \mathrel{\models\!\!\!\dashv} \psi$

  $\Sigma \models \varphi$ have same meanings as in PL
- φ is valid if every structure is a model of φ (we write $\models \varphi$)
- φ is satisfiable if it has a model
- Σ is satisfiable if there is A s.t. $A \models \varphi$ for all $\varphi \in \Sigma$.

Some more examples

① Let $L = \langle < \rangle$ be a language with a single binary relation symbol and let

$$A = \langle \mathbb{Q}, <^A \rangle$$

usual interpretation

Then we have

$$A \models \forall x \exists y (y < x)$$
$$A \not\models \exists y \forall x (y < x)$$
$$A \models \forall x \forall y (x < y \Rightarrow \exists z (x < z \wedge z < y))$$

One theme of FOL is that it can be annoying to refer to elements $x \in |A|$ unless you have a constant symbol that interprets as x.

② Consider the structures

multiplication

$L = \langle 0, \times, \leq \rangle$ $\qquad A = \langle \mathbb{R}, 0^A, \times^A, \leq^A \rangle$

$L' = \langle \times, \leq \rangle$ $\qquad A' = \langle \mathbb{R}, \times^{A'}, \leq^{A'} \rangle$

$L'' = \langle \times \rangle$ $\qquad A'' = \langle \mathbb{R}, \times^{A''} \rangle$

where all symbols have their usual interpretations.

<u>Question</u>: How can we express "A real x has a square root iff x is nonnegative" in these various languages?

in L: let φ be
$$\forall x(\exists y(y \times y = x) \Leftrightarrow 0 \leq x)$$

Then $A \models \varphi$

in L': trickier. Let φ' be
$$\exists z(\forall u(z \times u = z) \wedge \forall x(\exists y(y \times y = x) \Leftrightarrow z \leq x))$$

Then $A' \models \varphi'$ and the unique el't of $\mathbb{R}$ satisfying first clause is $z = 0$

Can we write an equivalent expression in L''?
(Don't think so).

<u>Tautologies</u>

- Fix a language L
- Recall that a sentence φ is <u>valid</u> if every A (in this language) has $A \models \varphi$.

- some sentences are valid because they have the "Boolean form" of a valid sentence in PL

- e.g. for any φ we know

$$\varphi \vee \neg\varphi$$

is valid: for any A, $Truth_A(\varphi \vee \neg\varphi) = 1$ iff $Truth_A(\varphi) = 1$ or $Truth_A(\neg\varphi) = 1$ which always holds.

- real reason is because $P \vee \neg P$ is a valid sentence in PL, and we have "substituted" φ for P.

- $Truth_A$ (in PL) ~~[illegible]~~ defined recursively on Boolean combos in same way $Truth_A$ (in FOL) is

$\neg \wedge \vee \Rightarrow \Leftrightarrow$

So any substitution of a valid PL sentence is valid in FOL

- More formally, if $\pi(P_0 \ldots P_{n-1})$ is a PL sentence in the variables $P_0, \ldots, P_{n-1}$ and $\varphi_0, \ldots, \varphi_{n-1}$ are FOL sentences we write $\pi(P_i/\varphi_i)$ for the FOL sentence obtained by replacing every instance of P_i with φ_i for $i < n$.

– if π is valid (in PL) we call $\pi(P_i/\varphi_i)$ a <u>tautology</u>

<u>Theorem</u> Tautologies are valid.

Pf: <u>clear</u>.

<u>examples</u> ① Let $\pi(P_0, P_1)$ be

$$(P_0 \Leftrightarrow P_1) \Rightarrow (\neg P_1 \Rightarrow \neg P_0)$$

Then π is valid (in PL). Let φ_0 be $1+1=0$ and let φ_1 be $\forall x(x \geq 0)$ then

$$((1+1=0) \Leftrightarrow \forall x(x \geq 0)) \Rightarrow (\neg(\forall x(x \geq 0)) \Rightarrow \neg(1+1)=0)$$

is a tautology (in the language $\langle 0, 1, +, \geq \rangle$) Hence this sentence is valid

② <u>Not all valid sentences are tautologies</u> e.g.

$$\forall x(x=x)$$

~~are~~ is valid. But ~~are~~ is not a tautologies

<u>More generally</u>: no sentence of form $\forall x \varphi$ / $\exists x \varphi$ is tautology

Basic Semantic Principles

- with PL we first defined deductive rules and then checked that for each rule, if $\vdash$ is replaced with $\models$ then we obtain a true statement. (soundness)
- with FOL we'll go a bit in opposite direction: prove some statements about $\models$ that we'll later "turn into" deductive rules

Substitution Lemma: Let s and t be closed terms and suppose φ is a formula w/ a single free variable v. Then:

$$s \approx t \models \varphi(v/s) \Leftrightarrow \varphi(v/t)$$

ex: Let $L = \langle \heartsuit, \ast, ☁, +, \leq \rangle$
where $\heartsuit, \ast, ☁$ are constants
Let s be $\heartsuit$
Let t be $\ast + \ast$
Let φ be $v \leq ☁$

Then Lemma says:

$$\heartsuit \approx \ast + \ast \models (\heartsuit \leq ☁) \Leftrightarrow (\ast + \ast \leq ☁)$$

Pf of Lemma

- only way to prove stuff about terms and fmlas is by induction on their construction. - but harder in FOL

- Suppose A is a struct and $A \models s \approx t$

- then there is $x \in |A|$ s.t. $s^A = t^A = x$

- Claim: if r is a term with at most the single variable v, then

$$r(v/s)^A = r(v/t)^A$$

Pf: induction on construction of r

- if r is a constant c
  then $r(v/s) = r(v/t) = c = r$
  and $r(v/s)^A = r(v/t)^A = c^A$

- if r is v
  then $r(v/s)$ is s
  $r(v/t)$ is t
  so that $r(v/s)^A = s^A = t^A = r(v/t)^A$

- sps r is $f(r_1, \ldots, r_n)$ and the claim holds for $r_1, \ldots, r_n$

then $r(v/s)$ is $f(r_1(v/s), \ldots, r_n(v/s))$

and $r(v/t)$ is $f(r_1(v/t), \dots, r_n(v/t))$

so that

$$r(v/s)^A = f^A(r_1(v/s)^A, \dots, r_n(v/s)^A)$$

$$r(v/t)^A = f^A(r_1(v/t)^A, \dots, r_n(v/t)^A)$$

but $r_i(v/t)^A = r_i(v/s)^A$ $\checkmark$ by IH

so $r(v/t)^A = r(v/s)^A$ $\checkmark$

Claim proved $\checkmark$

<u>Claim 2</u>: $A \models \varphi(v/s)$ iff $A \models \varphi(v/t)$

<u>Pf</u>: by induction on construction of φ.

- if φ is $\top$ or $\bot$ then $\varphi(v/s) = \varphi(v/t) = \varphi$, nothing to prove

- if φ is $r \approx r'$, where r, r' terms, then $\varphi(v/s)$ is $r(v/s) \approx r'(v/s)$ and $\varphi(v/t)$ is $r(v/t) \approx r'(v/t)$

Now:

$A \models r(v/s) \approx r'(v/s)$ ← $\varphi(v/s)$

iff $r(v/s)^A = r'(v/s)^A$

iff $r(v/t)^A = r'(v/t)^A$ ← by claim 1

iff $A \models r(v/t) \approx r'(v/t)$ ← $\varphi(v/t)$

• if φ is $R(r_1, \dots, r_n)$ where $r_1, \dots, r_n$ terms then $\varphi(v/s)$ is $R(r_1(v/s), \dots, r_n(v/s))$ and $\varphi(v/t)$ is $R(r_1(v/t), \dots, r_n(v/t))$

so that:

$$A \models R(r_1(v/s), \dots, r_n(v/s)) \leftarrow \varphi(v/s)$$

iff $R^A(r_1(v/s)^A, \dots, r_n(v/s)^A)$

claim 1 → iff $R^A(r_1(v/t)^A, \dots, r_n(v/t)^A)$

iff $A \models R(r_1(v/t), \dots, r_n(v/t)) \leftarrow \varphi(v/t)$

Sps claim holds for φ and ψ

then $(\varphi \wedge \psi)(v/s)$ is $\varphi(v/s) \wedge \psi(v/s)$ and " $\varphi \wedge \psi(v/t)$ " $\varphi(v/t) \wedge \psi(v/t)$

and we have $A \models \varphi(v/s) \wedge \psi(v/s)$

iff $A \models \varphi(v/s)$ and $A \models \psi(v/s)$

IH → iff $A \models \varphi(v/t)$ and $A \models \psi(v/t)$

iff $A \models \varphi(v/t) \wedge \psi(v/t)$

sim'ly for $\neg\varphi$, $\varphi \vee \psi$, $\varphi \Rightarrow \psi$, $\varphi \Leftrightarrow \psi$, $\forall u \varphi$ and $\exists u \varphi$. ✓

Claim is proved.

Claim 2 can also be stated

$$A \models \varphi(v/s) \Leftrightarrow \varphi(v/t) \checkmark$$

Lemma follows since A was arbitrary model of $s \approx t$ ✓

A word on quantifiers

Some useful notation: will write $\varphi(v)$ to mean φ is a fmla w/ a single free variable v, $\psi(u,v)$ means ψ has two free variables u and v etc.

Recall: • $A \models \exists v\, \varphi(v)$ iff there is $x \in |A|$ s.t. $(A,x) \models \varphi(^v/c_x)$

• $A \models \forall v\, \varphi(v)$ iff for every $x \in |A|$, we have $(A,x) \models \varphi(^v/c_x)$

But really: • $A \models \exists v\, \varphi(v)$ iff there is $x \in |A|$ s.t. "$\varphi(x)$ is true in A"

• $A \models \forall v\, \varphi(v)$ iff for every $x \in |A|$ "$\varphi(x)$ is true in A"

- have not defined what it means to "plug in" an arbitrary $x \in |A|$ for v; have defined $(A,x) \models \varphi(^v/c_x)$

- but when working in a specific structure can often cut out consideration of extra constants e.g. Sps $L = \langle R \rangle$ is a lang w/ a binary relation symbol and $A = \langle |A|, R^A \rangle$ is a structure in this lang.

Sps φ is $\forall u \exists v\, R(u,v)$

- Then $A \models \varphi$ iff for every $x \in |A|$ we have $(A, x) \models \exists v\, R(c_x, v)$
- iff for every $x \in |A|$ there is $y \in |A|$ s.t. $((A,x),y) \models R(c_x, c_y)$
- iff for every $x \in |A|$ there is $y \in |A|$ s.t. $R^{((A,x),y)}\left(c_x^{((A,x),y)}, c_y^{((A,x),y)}\right)$

iff $R^A(x, y)$

→ on HW or exam can cut out middle lines and say:

$$A \models \forall u \exists v\, R(u,v)$$

iff for every $x \in |A|$ there is $y \in |A|$ s.t. $R^A(x,y)$

- e.g. if $|A| = \mathbb{Z}$ and R^A is $<$ then

$A \models \forall u \exists v\, R(u,v)$ iff for every $n \in \mathbb{Z}$ there is $m \in \mathbb{Z}$ s.t. $n < m$ (true)