Compactness Theorem: Suppose Σ is a theory. Then Σ is satisfiable if (and only if) every finite subtheory $\Delta \subseteq \Sigma$ is satisfiable.

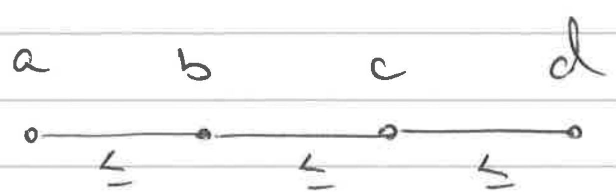
Def'n A **partial order** (p.o.) is a pair $(X, \leq)$ where X is a set and $\leq$ is a binary relation on X s.t.

① for every $x \in X$, $x \leq x$

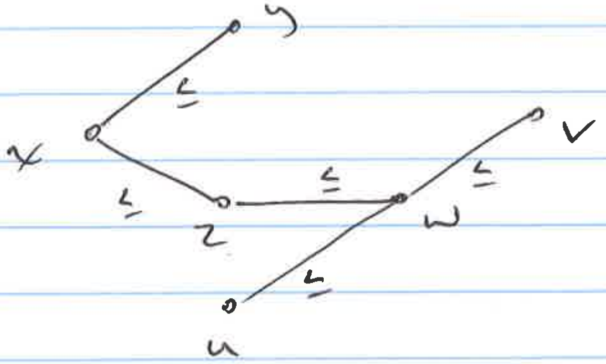
② if $x, y, z \in X$ and $x \leq y$ and $y \leq z$ then $x \leq z$

A **linear order** (l.o.) is a partial order $(X, \leq)$ also satisfying

③ for all $x, y \in X$, either $x \leq y$ or $y \leq x$



a linear order $a \leq b \leq c \leq d$

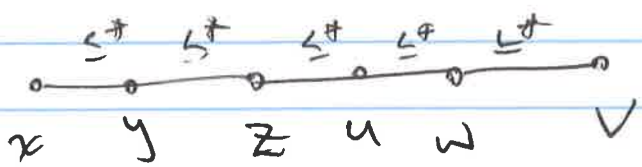


a partial order: not all elements comparable, e.g. $u \not\leq z$
$z \not\leq u$.

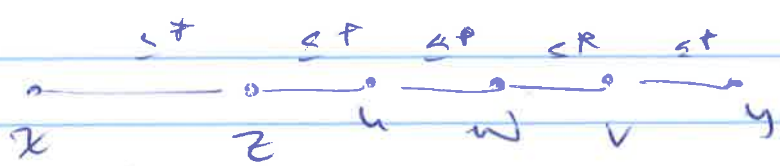
We say that a partial order $(X, \leq)$ can be <u>extended</u> to a linear order $(X, \leq^*)$ if for all $x, y \in X$ we have

$x \leq y$ implies $x \leq^* y$

an extension of the above p.o. to an l.o.



another one:



i.e. X is finite

Theorem every finite p.o. $(X, \leq)$ can be extended to a l.o. $(X, \leq^*)$

Pf: induction on $|X|$ ← cardinality of X

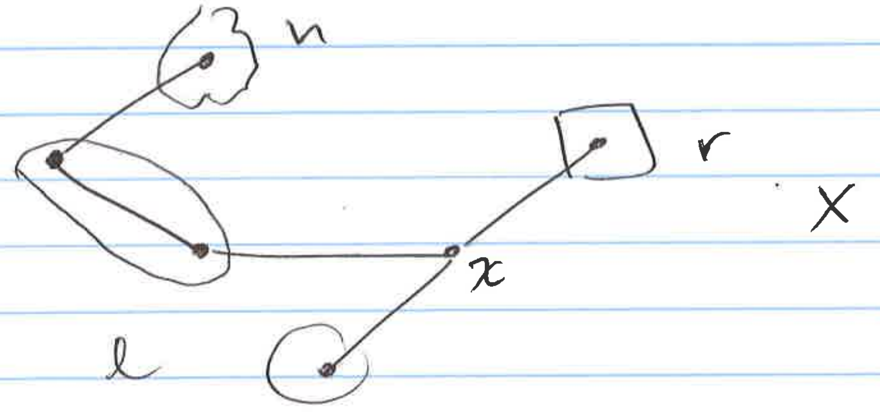
(BC) if $|X| = 1$ then $(X, \leq)$ is already a linear order

(IH) Suppose every p.o. $(X, \leq)$ with $|X| = n$ can be extended to a l.o. $(X, \leq^*)$

Now: · Suppose $(X, \leq)$ is a p.o. and $X = \{x_0, \ldots, x_{n-1}, x\}$ is of size $n+1$.

· we know that $\leq$ restricted to $\{x_0, \ldots x_{n-1}\}$ can be extended to a linear order $\leq^*$ on $\{x_0, \ldots, x_{n-1}\}$

· we extend this to a linear order $\leq^*$ on $\{x_0, \ldots, x_{n-1}, x_n\}$



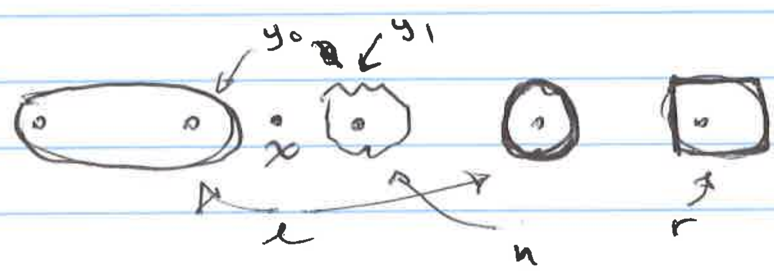
let $\ell = \{y \in X : y \leq x \text{ and } y \neq x\}$
$r = \{y \in X : x \leq y \text{ and } y \neq x\}$
$n = \{y \in X : x \not\leq y \text{ and } y \not\leq x\}$

Observe: • if $y \in \ell$ and $z \in r$ then $y \leq x$ and $x \leq z$ in original order $\leq$. Hence $y \leq z$.

• $\ell \cup r \cup n$ is linearly ordered by $\leq^*$

• let y_0 be max el't of ℓ under $\leq^*$ (if exists), and let y_1 be its successor

• extend $\leq^*$ to $\ell \cup r \cup n \cup \{x\}$ by setting $y_0 \leq^* x \leq y_1$



then $\leq^*$ still respects the original ordering $\leq$ since $x \leq^* z$ for all $z \in r$ and $y \leq^* x$ for all $y \in l$.

By induction any finite p.o. can be extended to an l.o.

<u>Theorem</u> every (countably) infinite p.o. $(X, \leq)$ can be extended to an l.o. $(X, \leq^*)$.

<u>Pf:</u>

- enumerate $X = \{x_0, x_1, \ldots\}$
- choose a bijection (relabeling) $f: X \times X \to (P_i)$
- denote $f(x_i, x_j)$ by $Q_{i,j}$
  - think of $Q_{i,j}$ as asserting "$x_i \leq^* x_j$"
- Let $\Sigma = \{Q_{i,j} : x_i \leq x_j \text{ in } X\}$
- Let $LO = \{Q_{i,i} : i \in \omega\}$
  $\cup \{(Q_{i,j} \wedge Q_{j,k}) \Rightarrow Q_{i,k} : i,j,k \in \omega\}$
  $\cup \{(Q_{i,j} \vee Q_{j,i}) : i,j \in \omega\}$

- Suppose we have a structure $A \models \Sigma \cup LO$
- define $\leq^*$ on X by

$$x_i \leq^* x_j \quad \text{iff} \quad A \models Q_{i,j}$$

<u>Claim</u>. $\leq^*$ is a linear order on X. extending $\leq$

<u>Pf</u>: • if $x_i \leq x_j$ then $Q_{i,j} \in \Sigma$
hence $A \models Q_{i,j}$ hence $x_i \leq^* x_j$

- $x_i \leq^* x_i$ for all i since $A \models Q_{i,i}$
- if $x_i \leq^* x_j$ and $x_j \leq^* x_k$ then $x_i \leq^* x_k$ since $A \models Q_{i,j} \wedge Q_{j,k} \rightarrow Q_{i,k}$
- for every i, j either $x_i \leq^* x_j$ or $x_j \leq^* x_i$ since $A \models Q_{i,j} \vee Q_{j,i}$ ✓

Question: does such A exist?

<u>Claim</u>, any finite subtheory $\Delta \subseteq \Sigma \cup LO$ is satisfiable.

<u>Pf</u>: write $\Delta = \Delta_0 \cup \Delta_1$
where $\Delta_0 \subseteq \Sigma$
$\Delta_1 \subseteq LO$

(M2)

let $X' = \{x_i \mid$ for some j, either $Q_{i,j}$ or $Q_{j,i}$ appears in a sentence in $\Delta\}$

- Then $(X', \leq)$ is a finite p.o.
- extend it to a l.o. $(X', \leq^*)$

- Define $A(Q_{i,j}) = 1$ iff $x_i, x_j \in X'$ and $x_i \leq^* x_j$

then $A \models \Delta_0$

<u>why</u>: if $Q_{i,j} \in \Delta_0$ then $x_i, x_j \in X'$ and $x_i \leq x_j$ hence $x_i \leq^* x_j$ hence $A \models Q_{i,j}$

and $A \models \Delta_1$

<u>why</u>, because $(X', \leq^*)$ is a linear order and the sentences in Δ_1 only refer to $x_i \in X'$

- hence $A \models \Delta$ ✓

→ Since Δ was arbitrary, <u>every</u> finite $\Delta \subseteq \Sigma \cup LO$ has a model

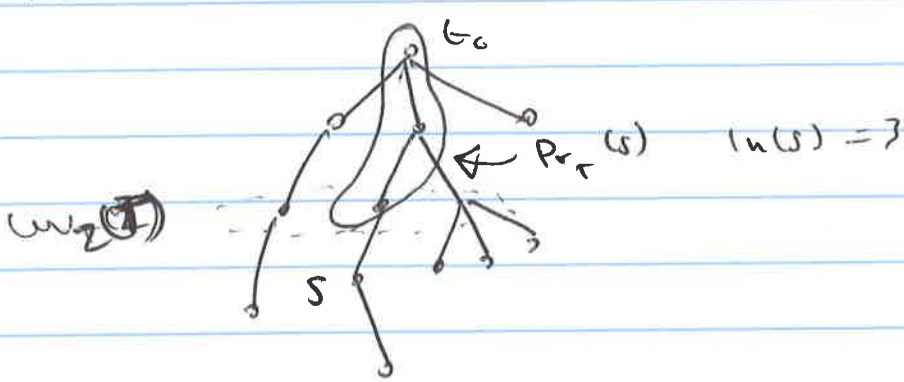
→ hence $\Sigma \cup LO$ does. Hence $(X, \leq)$ extends to $(X, \leq^*)$ ✓

König's infinity lemma

Def'n: a partial order $(T, \le)$ is called a tree if

- there is a unique minimal element $t_0 \in T$ (i.e. $t_0 \le s$ for all $s \in T$)
- for every $t \in T$, the set of predecessors, $Pr_T(s) = \{s \in T : s < t\}$ (strictly less) is a finite set linearly ordered by $\le$.

A tree:



- the length of a member $s \in T$, written $lh(s)$, is $|Pr_T(s)|$
- For every $n \in \omega$, the nth level of T is $Lev_n(T) = \{s \in T : lh(s) = n\}$

- T is finitely branching if $Lev_n(T)$ is finite for every n.

- a ~~branch~~ branch B is a maximal linearly ordered subset of T (i.e. B is linearly ordered by $\leq$ and if $t \notin B$ then $B \cup \{t\}$ is not linearly ordered by $\leq$)

<u>König infinity lemma</u>. Suppose $(T, \leq)$ is a (countable) finitely branching tree, and for every $n \in \omega$, $Lev_n(T) \neq \emptyset$. Then T has an infinite branch.

B

<u>Pf</u>: - To each $t \in T$ associate a prop'l variable P_t.

(45)

Consider the following collection of sentences

① For now, if $Lev_n(T) = \{t_1, \dots, t_k\}$ (finite, nonempty) consider $P_{t_1} \vee P_{t_2} \vee \dots \vee P_{t_k}$

② $\neg(P_{t_i} \wedge P_{t_j})$ for every $t_i, t_j \in Lev_n(T)$ with $t_i \neq t_j$

③ $P_t \Rightarrow P_s$ for every pair $s, t \in T$ with $s \leq t$

Let Σ denote the collection of all such sentences

<u>Claim</u> if Σ is satisfiable then T has an infinite branch

<u>Pf</u>: - Sps $A \models \Sigma$
- Let $B = \{t \in T \mid A \models P_t\}$ (i.e. $A(P_t) = 1$)

- then for every fixed n because A is a model of all sentences in ① and ② we have $A \models P_{t_i}$ for a <u>unique</u> $t_i \in Lev_n(T)$

- now suppose $s, t \in \mathcal{B}$ and $s \neq t$.
- WLOG assume $lh(s) < lh(t)$
  $\quad$ "n" $<$ "m"
- now: t has a unique predecessors' in $Lev_n(t)$
- we know $P_t \Rightarrow P_{s'} \in \Sigma$
- hence since $A \models P_t$
  $A \models P_t \Rightarrow P_{s'}$
  must have $A \models P_{s'}$

- but then it must be $P_{s'} = P_s$ hence $s = s'$

hence: $\mathcal{B}$ hits every level
- is linearly ordered
- is maximal

<u>Claim</u> every finite $\Delta \subseteq \Sigma$ is satisfiable.

"Δ only refers to t of length $\leq n$"

- if $\Delta \subseteq \Sigma$ is finite, there exists n s.t. if P_t occurs in a sentence in Δ then $lh(t) \leq n$

- pick $s \in Lev_n(T)$
- Define a structure by
  $A_s(P_t) = 1$ iff $t \leq s$ in T
- then $A_s \models \Delta$

the point:
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within $A \models D$

- hence satisfiable
- " Σ "
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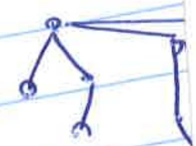
- hence
branch.

Finite branch



every level
but the