Ch. 2: Propositional Logic (PL)

- logical system in which statements are formed from propositional variables and connectives, e.g.

$$P_1 \wedge P_2$$
$$P_0 \Leftrightarrow (P_1 \vee P_2)$$

(think of each P_i as T/F)

- not able to refer to actual mathematical objects (numbers, functions, relations, etc.). For this need first-order logic (FOL).

- PL still useful as warmup for FOL: in FOL the variables P_i replaced by terms:
e.g. $(x \geq 0) \Leftrightarrow [(x > 0) \vee (x = 0)]$

- need to define the syntax and semantics of PL

syntax:
- symbols we can use
- which sequences of symbols are sentences
- which sequences of sentences are proofs

semantics: what it means for a sentence to be true given an interpretation of its symbols

Symbols in PL

$\top, \bot$ constants

$\neg, \vee, \wedge, \Rightarrow, \Leftrightarrow$ connectives

$($ $)$ $,$ non-logical symbols

$P_0, P_1, P_2, \ldots$ propositional variables (or parameters)

Let A = set of all symbols
$= \{\bot, \top, \neg, \vee, \wedge, \Rightarrow, \Leftrightarrow, (,), ,, P_0, P_1, \ldots\}$

- a string is an ordered sequence of symbols, e.g.
$$P_0)) \Rightarrow \bot \top \neg P_1$$

- Let $\mathcal{S}$ denote the set of all strings (really, think of $\mathcal{S}$ as $A^{<\omega}$)

- We recursively define a special subset of $\mathcal{S}$ called the set of sentences

- $\top$ and $\bot$ are sentences
- $P_0, P_1, \ldots$ are sentences
- if φ and χ are sentences then so are:

$(\neg, \varphi)$ (negation of φ)
$(\vee, \varphi, \chi)$ (disjunction of φ, χ)
$(\wedge, \varphi, \chi)$ (conjunction of φ, χ)
$(\Rightarrow, \varphi, \chi)$ (conditional)
$(\Leftrightarrow, \varphi, \chi)$ (biconditional)

For example:
$(\neg, P_1)$
$(\vee, (\Rightarrow, P_0, P_1), \top)$ are sentences

– More formally:
Define $S : \omega \to P(\mathcal{S})$ recursively:

- $S(0) = \{\bot, \top, P_0, P_1, \ldots\}$

- $S(n+1) = S(n) \cup \{(\neg, \varphi) \mid \varphi \in S(n)\}$
$\cup \{(\vee, \varphi, \chi) \mid \varphi, \chi \in S(n)\}$
$\cup \{(\wedge, \varphi, \chi) \mid \varphi, \chi \in S(n)\}$
$\cup \{(\Rightarrow, \varphi, \chi) \mid \varphi, \chi \in S(n)\}$
$\cup \{(\Leftrightarrow, \varphi, \chi) \mid \varphi, \chi \in S(n)\}$

- say a string φ is a sentence if there is n s.t. $\varphi \in S(n)$.

- e.g. $P_0 \in S(0)$
  $(\vee, (\Rightarrow, P_0, P_1), T) \in S(2)$

  are sentences.

↳ these two def'ns of sentence are (clearly) equivalent
↳ can you see how to define S using recursion theorem?

$\boxed{\text{Important}}$ Sentences are <u>uniquely readable</u>: i.e.

if $\varphi, \phi, \chi, \Psi$ are sentences then

equal as strings: same symbols in same order

- $(\neg, \varphi) = (\neg, \chi)$ implies $\varphi = \chi$
- $(\vee, \varphi, \phi) = (\vee, \chi, \Psi)$ implies
  $\varphi = \chi$
  $\phi = \Psi$
- above remains true if $\vee$ replaced by $\wedge, \Rightarrow, \Leftrightarrow$.

- e.g. only one way to decompose $(\wedge, P_0, (\neg, P_1))$ as a conjunction of two sentences: P_0 $(\neg, P_1)$
- ~~arbitrary~~ conjunctions of arbitrary strings not uniquely readable.
  e.g. could also decompose $(\wedge, P_0, (\neg, P_1))$ as the "conjunction" of
  $P_0, (\neg$ and $P_1)$
  but these are not sentences.

- if we were honest, would prove unique readability, but we won't
- still important: will be making def'ns that depend on unique readability

For example: can define complexity of a sentence φ recursively:

- $\perp, \top, P_0, P_1, \ldots$ have complexity 0
- if φ has complexity n, then $(\neg, \varphi)$ has complexity $n+1$
- if φ has comp n and χ has comp m then $(*, \varphi, \chi)$ has comp. $\max(n,m)+1$

  $\vee, \wedge, \Rightarrow, \Leftrightarrow$

e.g. $(\wedge, (\neg, P_1), P_2)$ has comp. 2.

(notice: def'n of comp. depends on unique readability)!

↳ we'll almost always write sentences using <u>traditional notation</u> ~~eg~~. i.e.

$$(\neg \varphi)$$
$$(\varphi \wedge \psi)$$
$$(\varphi \vee \psi)$$
$$(\varphi \Rightarrow \psi)$$
$$(\varphi \Leftrightarrow \psi)$$

↳ more readable, but be careful to insert parentheses:

$((P_1 \wedge P_2) \vee P_3)$ not the same as
$(P_1 \wedge (P_2 \vee P_3))$

↳ still have unique readability + notion of complexity for fmlas in traditional notation

Induction on the construction of ~~sentences~~ Sentences

Theorem let $R(\varphi)$ be a statement about the sentence φ.

if ① $R(\bot)$, $R(\top)$, $R(P_0)$, $R(P_1)$, ... all hold

② whenever $R(\varphi)$ and $R(\psi)$ both hold then $R(\neg\varphi)$, $R(\varphi \wedge \psi)$, $R(\varphi \vee \psi)$, $R(\varphi \Rightarrow \psi)$, $R(\varphi \Leftrightarrow \psi)$ all hold!

then $R(\varphi)$ holds for all sentences φ.

Pf: induct on the complexity of φ (You try)

ex: For a string φ, define $l(\varphi)$ = # of left parentheses in φ, $r(\varphi)$ = # of right parentheses

Prop'n in any sentence φ, $l(\varphi) = r(\varphi)$

Pf: induct on "$l(\varphi) = r(\varphi)$"

(BC) true for $\top, \bot, P_0, P_1, \ldots$
(no parentheses: $l = r = 0$)

(IH) Fix φ, χ and assume
$l(\varphi) = r(\varphi)$ and $l(\chi) = r(\chi)$

then $l((\neg\chi)) = l(\chi) + 1$
$\overset{\text{IH}}{=} r(\chi) + 1$
$= r((\neg\chi))$

and $l((\varphi \wedge \chi)) = l(\varphi) + l(\chi) + 1$
$\overset{\text{IH}}{=} r(\varphi) + r(\chi) + 1$
$= r((\varphi \wedge \chi))$

similarly for $\vee, \Rightarrow, \Leftrightarrow$

hence by induction $l(\varphi) = r(\varphi)$
for all sentences φ. ✓

Theorem the set of sentences is
countable (why?)

Pf: Let $A = \{\bot, \top, \vee, \wedge, \Rightarrow, \Leftrightarrow, P_0, P_1, \ldots,$
$(,)\}$

- there is a bijection $f : A \to \omega$

namely

$$f(\bot) = 0$$
$$f(\top) = 1$$
$$f(\vee) = 2$$
$$\vdots$$

- hence A is ctbl
- hence, since $\omega^{<\omega}$ is ctbl, $A^{<\omega}$ is ctbl (why?)
- can think of $A^{<\omega}$ as the <u>set of strings</u>
- Let Σ denote the set of sentences.
- then $\Sigma \subseteq A^{<\omega}$
- hence Σ is ctbl ✓

Semantics of PL

↳ so far: sentences just particular sequences of symbols.

↳ now: say what it means for sentence to be T/F (relative to truth of its parameters P_i)

- a <u>structure</u> is a function $A : \{P_n \mid n \in \omega\} \to \{0, 1\}$

- given a structure A, can extend to a function $\text{Truth}_A : \{\varphi \mid \varphi \text{ a sentence}\} \to \{0, 1\}$ recursively, as follows:

  - $\text{Truth}_A(\top) = 1$, $\text{Truth}_A(\bot) = 0$
  - $\text{Truth}_A(P_n) = A(P_n)$
  - $\text{Truth}_A(\neg\varphi) = 1$ iff $\text{Truth}_A(\varphi) = 0$
  - $\text{Truth}_A(\varphi \vee \psi) = 1$ iff $\text{Truth}_A(\varphi) = 1$ <u>or</u> $\text{Truth}_A(\psi) = 1$
  - $\text{Truth}_A(\varphi \wedge \psi) = 1$ iff $\text{Truth}_A(\varphi) = 1$ <u>and</u> $\text{Truth}_A(\psi) = 1$
  - $\text{Truth}_A(\varphi \Rightarrow \psi) = 1$ iff $\text{Truth}_A(\varphi) = 0$ or $\text{Truth}_A(\psi) = 1$
  - $\text{Truth}_A(\varphi \Leftrightarrow \psi) = 1$ iff $\text{Truth}_A(\varphi) = \text{Truth}_A(\psi)$

(note: def'n depends on unique readability)

<u>ex</u>: - Define a structure $A : \{P_n\} \to \{0,1\}$ by $A(P_n) = 1$ n even
$= 0$ n odd

- let φ_1 be $\neg P_1$
φ_2 be $(P_1 \Rightarrow P_2) \wedge P_3$
φ_3 be $P_1 \Leftrightarrow P_3$

- We have $A(P_1) = A(P_3) = 0$
$A(P_2) = 1$

- So $\mathrm{Truth}_A(\varphi_1) = 1$
$\mathrm{Truth}_A(\varphi_2) = 0$
$\mathrm{Truth}_A(\varphi_3) = 1$

Semantic consequence ($\models$) and equivalence ($\models\!\!\!|$)

- a ~~theorem~~ theory Σ is a set of ~~[illegible]~~ sentences

- e.g. $\Sigma_1 = \{\varphi_1, \varphi_2, \varphi_3\}$
$= \{\neg P_1, (P_1 \Rightarrow P_2) \wedge P_3, P_1 \Leftrightarrow P_3\}$

and $\Sigma_2 = \{\varphi_1, \varphi_3\}$
$= \{\neg P_1, P_1 \Leftrightarrow P_3\}$
are theories.

- a structure A is a <u>model</u> of φ if $\text{Truth}_A(\varphi) = 1$.
  ↳ we write $A \models \varphi$

- e.g. for A above we have
  $A \models \neg P_1$
  $A \not\models (P_1 \Rightarrow P_2) \wedge P_3$

- a structure A is a <u>model</u> for a theory Σ if $A \models \varphi$ for every $\varphi \in \Sigma$

- e.g. for A above we have
  $A \models \Sigma_2$
  $A \not\models \Sigma_1$

- For sentences φ, ψ we write
  $\varphi \models \psi$
("φ entails ψ") ("ψ is a semantic consequence of φ")
if every model of φ is a model of ψ.

ex let φ be $P_1 \Leftrightarrow P_2$
ψ be $P_1 \Rightarrow P_2$
then $\varphi \models \psi$

Why: to prove this, use a truth table

P_1	P_2	φ: $P_1 \Leftrightarrow P_2$	ψ: $P_1 \Rightarrow P_2$
1	1	1	1
1	0	0	0
0	1	0	1
0	0	1	1

Notice: any assignment of truth values to $\{P_1, P_2\}$ that makes φ true, also makes ψ true.

i.e. if A is a structure and $\text{Truth}_A(P_1 \Leftrightarrow P_2) = 1$ then $A(P_1) = A(P_2) = 0$ or $A(P_1) = A(P_2) = 1$

in either case $\text{Truth}_A(P_1 \Rightarrow P_2) = 1$

hence $\varphi \models \psi$.

- if Σ is a theory and φ is a sentence we write

$\Sigma \models \varphi$

if every model of Σ is a model of φ

- e.g. if $\Sigma = \{P_1, P_1 \Rightarrow P_2, \neg P_7\}$
  then $\Sigma \models P_2$

Why: if $A \models \Sigma$ then $A(P_1) = 1$
$\text{Truth}_A(P_1 \Rightarrow P_2) = 1$
hence $\text{Truth}_A(P_2) = 1$
i.e. $A \models P_2$

- φ and ψ are equivalent if
  $\varphi \models \psi$ and $\psi \models \varphi$. Write $\varphi \mathrel{\models\!\!\!\dashv} \psi$

e.g. if φ is $P_1 \Leftrightarrow P_2$
ψ is $\neg P_1 \Leftrightarrow \neg P_2$
then $\varphi \mathrel{\models\!\!\!\dashv} \psi$ (why?)
↳ truth table

- φ is valid if it is modeled by every structure
  e.g. $P_1 \vee \neg P_1$ is valid
  write $\models P_1 \vee \neg P_1$
  or $\emptyset \models P_1 \vee \neg P_1$

- φ is satisfiable if there is a model of φ
- Σ is satisfiable if there is a model of Σ

A deduction system

- above we define $\Sigma \models \varphi$
("φ follows semantically from Σ")

- now we will define $\Sigma \vdash \varphi$
("φ can be proved from Σ"
"φ follows syntactically from Σ")

The deduction rules

Sps Σ is a theory and φ, ψ, χ are sentences

$\boxed{\text{In}}$ $\boxed{\text{Out}}$

1. $\cancel{\emptyset} \vdash \top$
2. if $\Sigma \cup \{\varphi\} \vdash \psi$ and $\Sigma \cup \{\varphi\} \vdash \neg\psi$ then $\Sigma \vdash \neg\varphi$
3. $\{\varphi, \psi\} \vdash \varphi \wedge \psi$
4. $\{\varphi\} \vdash \varphi \vee \psi$ and $\{\psi\} \vdash \varphi \vee \psi$
5. if $\Sigma \cup \{\varphi\} \vdash \psi$ then $\Sigma \vdash \varphi \Rightarrow \psi$
6. $\{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi\} \vdash \varphi \Leftrightarrow \psi$

no bubbly

1'. $\{\bot\} \vdash \varphi$

2'. $\{\neg\neg\varphi\} \vdash \varphi$

3'. $\{\varphi \wedge \psi\} \vdash \varphi$ and $\{\varphi \wedge \psi\} \vdash \psi$

4'. if $\Sigma \cup \{\varphi\} \vdash \chi$ and $\Sigma \cup \{\psi\} \vdash \chi$ then $\Sigma \cup \{\psi \vee \varphi\} \vdash \chi$

5'. $\{\varphi, \varphi \Rightarrow \psi\} \vdash \psi$

6'. $\{\varphi \Leftrightarrow \psi\} \vdash \varphi \Rightarrow \psi$ and $\{\varphi \Leftrightarrow \psi\} \vdash \psi \Rightarrow \varphi$

R1 if $\Delta \subseteq \Sigma$ and $\Delta \vdash \varphi$ then $\Sigma \vdash \varphi$

R2 if $\Sigma \vdash \varphi_i$ for all $i < n$ and $\{\varphi_i \mid i < n\} \vdash \psi$ then $\Sigma \vdash \psi$

R3 if $\varphi \in \Sigma$ then $\Sigma \vdash \varphi$.

Examples of deductions (before formal defn)

ex $\emptyset \vdash \neg\neg P_1 \Rightarrow P_1$

Why: ① $\{\neg\neg P_1\} \vdash P_1$ by 2′ $\neg$-elt
i.e. $\emptyset \cup \{\neg\neg P_1\} \vdash P_1$
so: ② $\emptyset \vdash \neg\neg P_1 \Rightarrow P_1$ by 5 $\Rightarrow$-in

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ex let $\Sigma = \{P_1, P_1 \Leftrightarrow P_2\}$
then $\Sigma \vdash P_2$

Why
① $\Sigma \vdash P_1 \Leftrightarrow P_2$ by R3
② $\{P_1 \Leftrightarrow P_2\} \vdash P_1 \Rightarrow P_2$ by 6′ $\Leftrightarrow$-elt
③ $\Sigma \vdash P_1 \Rightarrow P_2$ by R2, line ②
④ $\Sigma \vdash P_1$ by R3
⑤ $\{P_1, P_1 \Rightarrow P_2\} \vdash P_2$ by 5′ $\Rightarrow$-elt
⑥ $\Sigma \vdash P_2$ by R2, lines ③ ④, ⑤

Some of the deduction rules are redundant

ex Suppose Σ is a theory and $\varphi \in \Sigma$.

Then:

1. $\{\varphi\} \vdash \varphi \wedge \varphi$ — by 3 $\wedge$-in
2. $\{\varphi \wedge \varphi\} \vdash \varphi$ — by 3′ $\wedge$-out
3. $\{\varphi\} \vdash \varphi$ — by lines ①, ② and R2
4. $\Sigma \vdash \varphi$ — by R1 and line ③

- This shows that the rule R3 is redundant: follows from other rules.
- in any proof involving R3 could replace with a version of above (but would make for a longer proof)

- The above are examples of formal deduction (or justification)

- in general a justification is a sequence

$$((\Pi_0, \varphi_0), \ldots, (\Pi_n, \varphi_n))$$

where each Π_i is a theory
φ_i is a sentence
and $\Pi_i \vdash \varphi_i$ follows from $((\Pi_0, \varphi_0), \ldots, (\Pi_{i-1}, \varphi_{i-1}))$ and a deduction rule

single deduction every time!

Soundness

- Fix a theory Σ and sentence φ
  Q: what is the relation between

$$\Sigma \vdash \varphi \qquad \text{(purely syntactic)}$$

and

$$\Sigma \models \varphi \qquad \text{(semantic)}$$

- turns out these notions are equivalent, i.e.

$$\Sigma \vdash \varphi \quad \text{iff} \quad \Sigma \models \varphi$$

- the forward direction $\Rightarrow$ is called <u>soundness</u> of PL (if we can prove it from Σ, then it's true in every model of Σ)

surprising! - $\Leftarrow$ called <u>completeness</u> of PL (if it's true in every model, then we can actually prove it!)

<u>Thm</u> (Soundness of PL) Fix a theory Σ and sentence φ. If $\Sigma \vdash \varphi$ then $\Sigma \models \varphi$.

<u>Pf:</u> "obvious" (ahem)

- idea: in every deduction rule, can prove we can replace $\vdash$ with $\models$; theorem follows immediately (why?)
  - e.g. consider "$\{\varphi, \chi\} \vdash \varphi \wedge \chi$"
  - fix a structure A with $A \models \{\varphi, \chi\}$

    then $Truth_A(\varphi) = Truth_A(\chi) = 1$

    hence $Truth_A(\varphi \wedge \chi) = 1$

    hence $A \models \varphi \wedge \chi$

    since A arbitrary, $\{\varphi, \chi\} \models \varphi \wedge \chi$.

  - e.g. also consider

    "if $\Sigma \cup \{\varphi\} \vdash \chi$ then $\Sigma \vdash \varphi \Rightarrow \chi$"

  - let's prove

    "if $\Sigma \cup \{\varphi\} \models \chi$ then $\Sigma \models \varphi \Rightarrow \chi$"

<u>Pf:</u> - assume $\Sigma \cup \{\varphi\} \models \chi$. WTS $\Sigma \models \varphi \Rightarrow \chi$

- Fix $A \models \Sigma$
- if $Truth_A(\varphi) = 0$ then

  $Truth_A(\varphi \Rightarrow \chi) = 1$ hence $A \models \varphi \Rightarrow \chi$
- if $Truth_A(\varphi) = 1$

  then $A \models \Sigma \cup \{\varphi\}$ hence by assumption

  $A \models \chi$, i.e. $Truth_A(\chi) = 1$.
- hence $Truth_A(\varphi \Rightarrow \chi) = 1$

  i.e. $A \models \varphi \Rightarrow \chi$

↳ In all cases $A \models \varphi \Rightarrow \psi$

- since $A \models \Sigma$ was arbitrary,

[redacted] $\Sigma \models \varphi \Rightarrow \psi$, as desired. ✓

- and likewise for the other deduction rules: we can turn every rule into a "$\models$ version"

- then if we have some proof that $\Sigma \vdash \varphi$, e.g.

$\Gamma_0 \vdash \varphi_0$ because (...)
$\Gamma_1 \vdash \varphi_1$ because (...)
⋮
$\Gamma_{n-1} \vdash \varphi_{n-1}$ because (...)
$\Sigma \vdash \varphi$ because (...)

deduction rules w/ $\vdash$

can turn it into proof that $\Sigma \models \varphi$

$\Gamma_0 \models \varphi_0$ because (...)
because (...)
⋮
$\Gamma_{n-1} \models \varphi_n$
$\Sigma \models \varphi$ because (...)

corresponding deduction rules with $\models$.

✓

bottom line: deduction rules play nicely w/ semantic notion of truth (don't break it)

↳ converse to soundness is completeness:
if $\Sigma \models \varphi$ then $\Sigma \vdash \varphi$

↳ idea is: our deduction rules are "sufficient" to turn any (informal) proof that $\Sigma \models \varphi$ into a formal deduction that $\Sigma \vdash \varphi$.

↳ this is not how we will prove completeness (would involve quantifying over all informal arguments that $\Sigma \models \varphi$)

~~we~~ need some preliminary results:

<u>Thm</u> (Finiteness of deduction).
If $\Sigma \vdash \varphi$ then there is a finite subset $\Delta \subseteq \Sigma$ s.t. $\Delta \vdash \varphi$.

<u>Pf</u>: "obvious" (ahem) because justifications are finite, and in any deductive step can refer to only finitely many sentences from previous steps. —

<u>really</u>: we induct on the statement "if $\Sigma \vdash \varphi$ has a justification of length n, then there is a finite $\Delta \subseteq \Sigma$ s.t. $\Delta \vdash \varphi$"

Observe: there are two types of deduction rules

absolute rules: e.g. $\{\varphi, \psi\} \vdash \varphi \wedge \psi$

conditional rules e.g. if $\Sigma \cup \{\varphi\} \vdash \psi$ then $\Sigma \vdash \varphi \Rightarrow \psi$

- all absolute rules have finite sets on left of $\vdash$.
- no conditional rule can be used as first step in justification, except R3

let us assume our deduction does not use R3

now we induct

(BC) Sps $\Sigma \vdash \varphi$ has a justification of length 1, then is given by absolute rule so Σ is finite ✓. (or R3, which we can dispense with)

(IH) Assume in any justification

$\Pi_0 \vdash \varphi_0$

$\vdots$

$\Pi_{n-1} \vdash \varphi_{n-1}$

of length n there exist finite $\Delta_k \subseteq \Pi_k$ s.t. $\Delta_k \vdash \varphi_k$ for every $k < n$.

Now suppose

$\Gamma_0 \vdash \varphi_0$

$\vdots$

$\Gamma_{n-1} \vdash \varphi_{n-1}$

$\Sigma \vdash \varphi$ is a justification of length $n+1$

- last step is justified by single deduction rule.
- if this is an absolute rule, then Σ is finite, so done.
- So suppose last step given by a conditional rule: $\neg$-in, $\Rightarrow$-in, $\vee$-out, R1, R2, R3

- e.g. if $\neg$-in, then $\varphi = \neg\psi$ for some ψ and two previous lines in the justification are

$\Sigma \cup \{\psi\} \vdash \chi$

$\Sigma \cup \{\neg\psi\} \vdash \chi$

- by induction, there is finite $\Delta \subseteq \Sigma$ s.t. $\Delta \cup \{\psi\} \vdash \chi$

$\Delta \cup \{\neg\psi\} \vdash \chi$

- hence $\Delta \vdash \neg\psi$ by $\neg$-in i.e. $\Delta \vdash \varphi$
- but $\Delta \subseteq \Sigma$ is finite, as desired.

- Similar arg works for other conditional deduction rules ✓

↳ we'll need finiteness of deductions to prove completeness of PL

↳ some more terminology:

- ↳ if $\Sigma \vdash \varphi$ we say φ is a <u>theorem</u> of Σ
- ↳ Σ is <u>consistent</u> if $\bot$ is <u>not</u> a theorem of Σ
- ↳ (Σ is <u>inconsistent</u> if $\Sigma \vdash \bot$)
- ↳ Σ is <u>complete</u> if for every sentence φ, either $\Sigma \vdash \varphi$ or $\Sigma \vdash \neg\varphi$.

↳ complete theories are "strong" — they determine truth or falsity of every φ

<u>Warning</u>: "complete" here has entirely different meaning than "completeness of PL"

"Σ is complete" means $\Sigma \vdash \varphi$ or $\Sigma \vdash \neg\varphi$ for every φ

"Propositional logic is complete" means for every theory Σ and sentence φ, if $\Sigma \models \varphi$ then $\Sigma \vdash$

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*important corollary to soundness:

<u>Cor</u> if Σ has a model, then Σ is consistent
(equiv. if Σ is inconsistent, Σ has no model)

<u>Pf</u>: if $\Sigma \vdash \bot$ and $A \models \Sigma$ then $A \models \bot$
i.e. $truth_A(\bot) = 1$
contradiction

Converse is key to proof of completeness

<u>Theorem</u> TFAE:

1. $\Sigma \vdash \bot$ (Σ is inconsistent)
2. for every φ, $\Sigma \vdash \varphi$
3. for some φ, $\Sigma \vdash \varphi$ and $\Sigma \vdash \neg\varphi$.

<u>Pf</u>: <u>1 ⟹ 2</u> - Suppose $\Sigma \vdash \bot$ and fix a sentence φ
- then $\{\bot\} \vdash \varphi$ by $\bot$-out
- hence $\Sigma \vdash \varphi$ by R2.

<u>2 ⟹ 3</u> Obvious: fix any sentence ψ. Then by 2, $\Sigma \vdash \psi$ and $\Sigma \vdash \neg\psi$ ✓

<u>3 ⟹ 1</u>. - Suppose there is φ s.t. $\Sigma \vdash \varphi$ and $\Sigma \vdash \neg\varphi$.
- then $\Sigma \cup \{\neg\bot\} \vdash \varphi$ by R1
- and $\Sigma \cup \{\neg\bot\} \vdash \neg\varphi$ by R1
- hence $\Sigma \vdash \neg\neg\bot$ by $\neg$-in
- and $\{\neg\neg\bot\} \vdash \bot$ $\neg$-out
- hence $\Sigma \vdash \bot$ by R2

✓

→ don't get too caught up by particular deductions we needed to prove this theorem, which are a bit awkward

↳ point is: we just need some deduction system whose rules allow us to prove this

↳ our system works, but isn't unique in this regard.

→ Theorem tells us: inconsistent theories are uninteresting (they prove everything).

→ in particular, inconsistent theories have no models (i.e. are unsatisfiable)

- if Σ is inconsistent then $\Sigma \vdash P_0$ and $\Sigma \vdash \neg P_0$
- hence by soundness $\Sigma \models P_0$ and $\Sigma \models \neg P_0$
- i.e. every model $A \models \Sigma_0$ must also have $A \models P_0$, $A \models \neg P_0$
- of course, there are no such structures A
- hence Σ is unsatisfiable

↳ another way to phrase this: if Σ has a model, then Σ is consistent!

Ex: $\{P_0, P_1, P_2, P_3\}$ is consistent

Note: hard to show directly!

→ what about consistent theories?
→ next goal is to show every consistent Σ does have a model.
→ need some preliminaries first

Lemma Suppose Σ is consistent and φ is a sentence.
Then either $\Sigma \cup \{\varphi\}$
possibly both → or $\Sigma \cup \{\neg\varphi\}$ is consistent

Pf. - Suppose not
- Then both $\Sigma \cup \{\varphi\}$ and $\Sigma \cup \{\neg\varphi\}$ are inconsistent
- hence by theorem
$\Sigma \cup \{\varphi\} \vdash P_0$ and $\Sigma \cup \{\varphi\} \vdash \neg P_0$
and $\Sigma \cup \{\neg\varphi\} \vdash P_0$ and $\Sigma \cup \{\neg\varphi\} \vdash \neg P_0$

↓ by $\neg$-in ↓ by $\neg$-in

$\Sigma \vdash \neg\varphi$ $\Sigma \vdash \neg\neg\varphi$

⇒ Σ is inconsistent, by theorem.
- Contradiction, since we assumed Σ was consistent ✓

Complete

Extension Lemma: if Σ is consistent, then there is a consistent, complete theory Σ' with $\Sigma \subseteq \Sigma'$.

Pf: sps Σ is consistent
- we'll actually build Σ' so that for every φ, either $\varphi \in \Sigma'$ or $\neg\varphi \in \Sigma'$
  - such a Σ' is clearly complete, by R3.

↳ by a previous result, the set of sentences is countable, so we may enumerate them:

$$\{\varphi_0, \varphi_1, \ldots\}$$

↳ inductively define a sequence of theories Σ_n

- $\Sigma_0 = \Sigma$
- $\Sigma_{n+1} = \Sigma_n \cup \{\varphi_n\}$ if this is consistent

  $= \Sigma_n \cup \{\neg\varphi_n\}$ if $\Sigma_n \cup \{\varphi_n\}$ is inconsistent

- $\Sigma_0 = \Sigma$ is consistent by hypothesis
- and if Σ_n is consistent then either $\Sigma_n \cup \{\varphi_n\}$ or $\Sigma_n \cup \{\neg\varphi_n\}$ is consistent by the model extension lemma.

- hence Σ_n is consistent for all n. We have $\Sigma_0 \subseteq \Sigma_1 \subseteq \ldots$
- let $\Sigma' = \bigcup \Sigma_n$

- Clearly Σ' is complete, since $\varphi \in \Sigma'$ or $\neg\varphi \in \Sigma'$ for every φ.

- but Σ' is also consistent

  <u>Why</u>: - if not, then $\Sigma' \vdash \bot$
  - by finiteness of deduction there would be a finite subtheory $\Delta \subseteq \Sigma$ s.t. $\Delta \vdash \bot$
  - but if Δ is finite then for some n, $\Delta \subseteq \Sigma_n$.
  - hence $\Sigma_n \vdash \bot$ by R1
  - contradiction

hence Σ' is consistent and complete and $\Sigma \subseteq \Sigma'$ ✓

<u>ex</u>: - Consider $\Sigma = \{P_0, P_1 \Rightarrow P_2, P_5\}$
- Σ is consistent (why?)
- Can extend

Σ

$\Sigma \cup \{P_1\}$ $\qquad$ $\Sigma \cup \{\neg P_1\}$ $\qquad$ both consistent extensions (why?)

$\Sigma \cup \{P_1\} \cup \{\neg P_2\}$ not consistent

$\Sigma \cup \{P_1\} \cup \{P_2\}$ consistent

$\Sigma \cup \{\neg P_1\} \cup \{P_2\}$ $\qquad$ $\Sigma \cup \{\neg P_1\} \cup \{\neg P_2\}$ both consistent

⋮ $\qquad$ ⋮

eventually get a complete consistent extension of Σ.

<u>Recall</u>: Soundness gives us:
if Σ has a model
then Σ consistent

here is the converse:

<u>Theorem</u> If Σ is consistent, then Σ has a model.

<u>Pf</u>: – Suppose Σ is consistent.
– there is $\Sigma' \supseteq \Sigma$ s.t. for every φ, either $\varphi \in \Sigma'$ or $\neg\varphi \in \Sigma'$ and Σ' is consistent.

<u>Claim 1</u>: For every φ, <u>exactly one</u> <u>of</u> φ and $\neg\varphi$ is in Σ'

<u>Pf</u>: o.w. Σ' is inconsistent (why?)

<u>Claim 2</u>: For every φ, $\Sigma' \vdash \varphi$ iff $\varphi \in \Sigma'$

<u>Pf</u>: $\Leftarrow$-direction given by R3
($\Rightarrow$) Assume $\Sigma' \vdash \varphi$. Either $\varphi \in \Sigma'$ or $\neg\varphi \in \Sigma'$. If $\neg\varphi \in \Sigma'$ then $\Sigma' \vdash \neg\varphi$ and Σ' is inconsistent, a contradiction. Hence $\varphi \in \Sigma'$

Define a structure $A : \{P_i\} \to \{0,1\}$ by

$$A(P_i) = 1 \quad \text{if } P_i \in \Sigma'$$
$$A(P_i) = 0 \quad \text{if } P_i \notin \Sigma'$$

Claim 3 $\text{Truth}_A(\varphi) = 1$ iff $\varphi \in \Sigma'$

Pf: by induction on construction of φ.

(BC) • $\text{Truth}_A(P_i) = A(P_i) = 1$ iff $P_i \in \Sigma'$
by def'n of A

• $\text{Truth}_A(\top) = 1$
and $\top \in \Sigma'$ since $\Sigma' \vdash \top$ (by

• $\text{Truth}_A(\bot) = 0$
and $\bot \notin \Sigma'$ since Σ' is
consistent using R3 here)

(IH) Sps φ, ψ are sentences
and $\text{Truth}_A(\varphi) = 1$ iff $\varphi \in \Sigma'$
$\text{Truth}_A(\psi) = 1$ iff $\psi \in \Sigma'$

Then $\text{Truth}_A(\neg\varphi) = 1$ iff
$\text{Truth}_A(\varphi) = 0$ iff
$\varphi \notin \Sigma'$ (by IH)
iff $\neg\varphi \in \Sigma'$ (by Claim 1)

Now need to check $\text{Truth}_A(\varphi * \psi) = 1$
iff $\varphi * \psi \in \Sigma'$, where $*$ is $\wedge$ $\vee$ $\Rightarrow$
or $\Leftrightarrow$

We check for $\wedge$:

$\text{Truth}_A(\varphi \wedge \chi) = 1$ iff
$\text{Truth}_A(\varphi) = 1$ and $\text{Truth}_A(\chi) = 1$ iff
$\varphi \in \Sigma'$ and $\chi \in \Sigma'$ (by IH) iff
$\{\varphi, \chi\} \subseteq \Sigma'$

<u>Claim</u>: $\{\varphi, \chi\} \subseteq \Sigma'$ iff $\Sigma' \vdash \varphi \wedge \chi$

<u>Pf</u>: ($\Rightarrow$) Assume $\{\varphi, \chi\} \subseteq \Sigma'$
then $\Sigma' \vdash \varphi \wedge \chi$ by $\wedge$-out, R1

in each case use el and in rule for each direction

($\Leftarrow$) Assume $\Sigma' \vdash \varphi \wedge \chi$
then $\Sigma' \vdash \varphi$ and $\Sigma' \vdash \chi$ by $\wedge$-in, R2
hence $\varphi \in \Sigma'$ and $\chi \in \Sigma'$ by claim 2, i.e. $\{\varphi, \chi\} \subseteq \Sigma'$ ✓

So we continue
iff $\Sigma' \vdash \varphi \wedge \chi$
iff $\varphi \wedge \chi \in \Sigma'$ by claim 2 ✓

Similarly for $\vee, \Rightarrow, \Leftrightarrow$.
Hence by induction $\text{Truth}_A(\varphi) = 1$ iff $\varphi \in \Sigma'$ ✓

It follows that $A \vDash \Sigma'$
hence $A \vDash \Sigma$ ✓

Proof not crazy: given consistent Σ, extend to consistent Σ' containing P_i or $\neg P_i$ for every i
Define $A(P_i) = 1$ iff $P_i \in \Sigma'$
Then $A \models \Sigma'$ (and hence $A \models \Sigma$)

But theorem is powerful:

Theorem (Completeness of PL). ~~[illegible]~~
Suppose Σ is a theory and φ a sentence. Then if $\Sigma \models \varphi$, then $\Sigma \vdash \varphi$.

Pf: - By contrapositive
- closure statement is trivial if Σ is inconsistent, so assume Σ is consistent and $\Sigma \nvdash \varphi$.

Claim $\Sigma \cup \{\neg\varphi\}$ is consistent
why: if $\Sigma \cup \{\neg\varphi\}$ is inconsistent
then $\Sigma \cup \{\neg\varphi\} \vdash \chi$
$\Sigma \cup \{\neg\varphi\} \vdash \neg\chi$ for some χ
hence $\Sigma \vdash \neg\neg\varphi$ by $\neg$-in
hence $\Sigma \vdash \varphi$ by $\neg$-out
R2.

contradiction

- hence $\Sigma \cup \{\neg \varphi\}$ is consistent as claimed.

- hence there is a model $A \models \Sigma \cup \{\neg\varphi\}$

- hence $A \models \Sigma$
  and $A \not\models \varphi$

- but then $\Sigma \not\models \varphi$, theorem is proved. ✓

__eg__: Let $\Sigma = \{P_0, P_0 \Rightarrow P_1, P_1 \Rightarrow P_2, P_2 \Rightarrow P_3, \ldots\}$
Then $\Sigma \vdash P_3$

__Pf__: We prove $\Sigma \models P_3$
- if $A \models \Sigma$, then $\text{Truth}_A(P_0) = 1$
  ~~hence~~ $\text{Truth}_A(P_1) = 1$ since $\text{Truth}_A(P_0 \Rightarrow P_1) = 1$.
  hence $\text{Truth}_A(P_2) = 1$ since $\text{Truth}_A(P_1 \Rightarrow P_2) = 1$
  hence $\text{Truth}_A(P_3) = 1$

~~hence~~ so since A arbitrary $\Sigma \vdash P_3$

__point__: our deduction system powerful enough to turn these proofs into formal deductions.

Here is a useful corollary of model existence

<u>Compactness Theorem</u>: Suppose Σ is a theory and every finite subtheory $\Delta \subseteq \Sigma$ has a model. Then Σ has a model

<u>Pf.</u> – Toward a contradiction, suppose Σ has no model

– then by Completeness theorem Σ is inconsistent.

↳ why: if Σ consistent then Σ has a model

– hence $\Sigma \vdash \bot$

– by finiteness of deduction there is a finite subtheory $\Delta \subseteq \Sigma$ s.t. $\Delta \vdash \bot$

– but then Δ has no model, a contradiction

– hence Σ has a model ✓