21-300 Basic Logic

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syllabus w/ rough schedule posted

Office Hours

Monday: 10:30 – 11:30 in WEH 7201
- "Structured" office hour
- more like a recitation: may present practice problems some weeks

Wednesday: 10:30 – noon in WEH 7128 (my office)
- standard office hour; just answering HW q's etc.

Prof Schimmerling's Office Hours:
Monday: 4:30 – 5:30 in WEH 8201 (structured)
Wednesday: 4:30 – 5:30 in WEH 7125 (his office) (standard)

Grading

HW: 20% (weekly)
Midterm 1: 25%
Midterm 2: 25%
Final: 30%

- Tests individually curved
- Final cutoffs not more harsh than 90 - 100 = A, 80 - 90 = B etc.

Textbook: Schwmmerling

- linked on Canvas page
- will cover Ch. 1-4

↳ HW will be posted on Canvas
↳ First midterm is Oct. 12

Overview of Class

↳ introduction to mathematical logic
↳ Big themes: relationships between
(1) mathematical syntax (what is written) and mathematical semantics (what is meant)

"A word is a chest for its rich contents, as a man is a cloak for his soul."

(2) what we can prove and what is true.

↳ will study propositional logic and first-order logic.
↳ there are logical systems in which we can write formal statements

e.g. $(\forall x \in \mathbb{R})\ x^2 \geq 0$

and carry out formal proofs

e.g. $\rightarrow (\forall x \in \mathbb{R})\ x^2 \geq 0$
$\rightarrow -5 \in \mathbb{R}$
$\rightarrow (-5)^2 \geq 0$

→ these systems give us a formal model of mathematical practice ("from a set of axioms, deduce new theorems by way of proofs.")

Amazing fact: in this formal setting, notion of proof is entirely syntactic, i.e. depends only on symbols on page and certain deduction rules

"Mathematics is a game played according to certain simple rules with meaningless marks on paper."

→ from "outside" of these systems can define what it means for a statement to be true and then investigate the q's:

① if we can prove a statement S, is it true?

② if a statement is true, can we prove it?

Before that, some review

Set-theoretic Background

- Informally, a *set* is a collection of objects ("set" = "collection" = "family")
- e.g.

$$A = \{*, ♡, \Delta\}$$

$$E = \{0, 2, 4, 6, \ldots\}$$

$$= \{x \in \mathbb{N} \mid x \text{ is a multiple of } 2\}$$

$$= \{x \in \mathbb{N} \mid (\exists k \in \mathbb{N})\ x = 2k\}$$

- last two lines examples of set-builder notation, defining the set E of even natural #'s

- $a \in X$ means "a is an element of X"
- $a \notin X$ means "a is not an el't of X"

  e.g. $♡ \in A$, $2 \in E$, but $♡ \notin E$

- sets are determined by their el'ts, order, repetition irrelevant

  e.g. $\{1,2,3\} = \{2,1,3\} = \{1,1,2,3\}$

- sets may contain other sets

  e.g. $X = \{\{1,2\}, \{3,4\}\}$ is a set w/ two elements: $\{1,2\}$ and $\{3,4\}$

- different from $Y = \{1,2,3,4\}$

- if A, B are sets, the <u>union</u> of A, B is
$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$
- e.g. if $A = \{1,2\}$
$B = \{2,5,6\}$
then $A \cup B = \{1,2,5,6\}$

- if S is a collection of sets then the <u>union over S</u> is:
$$\bigcup S = \{x \mid \text{there exists } A \in S \text{ with } x \in A\}$$

- e.g. if $S = \{\{1,2\}, \{3,4\}, \{5,6\}, \ldots\}$
then
$$\bigcup S = \{1,2\} \cup \{3,4\} \cup \{5,6\} \cup \cdots$$
$$= \{1,2,3,4,5,\ldots\}$$

- hence we can also write $A \cup B$ as $\bigcup \{A, B\}$

- if A, B are sets, the <u>intersection</u> of A, B is
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

- if S is a collection of sets then we define

$$\bigcap S = \{x \mid \text{for every } A \in S \text{ we have } x \in A\}$$

- e.g. if $S = \{\{1,2,3,4\}, \{2,4,6,8\}, \{0,2,4\}\}$

then

$$\bigcap S = \{1,2,3,4\} \cap \{2,4,6,8\} \cap \{0,2,4\}$$
$$= \{2,4\}$$

- hence we can also write $\bigcap\{A,B\}$ for $A \cap B$.

- the <u>empty set</u> is the unique set with no el'ts
- denoted $\{\}$ or $\emptyset$

- if A, B are sets then A is a <u>subset</u> of B if for every $x \in A$ we have $x \in B$
- we write $A \subseteq B$

- e.g. $\{1,3\} \subseteq \{1,2,3,4\}$
but $\{1,5\} \not\subseteq \{1,2,3,4\}$

- For any set A we have
  $\emptyset \subseteq A$ and $A \subseteq A$.

- the powerset of A, written $P(A)$, is the set of all subsets of A

- e.g. if $A = \{1,2,3\}$ then

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Fact: If A is a finite set with n elements then A has 2^n subsets, i.e. $P(A)$ has 2^n elements

Some important sets

ω = set of natural #'s
$= \{0,1,2,3,\ldots\}$
$= \mathbb{N}$

$\mathbb{Z}$ = set of integers
$= \{\ldots,-2,-1,0,1,2,\ldots\}$

$\mathbb{Q} = \{\frac{m}{n} \mid m,n \in \mathbb{Z} \text{ and } n \neq 0\}$
$\mathbb{R}$ = set of real #'s.

Cartesian Products

- if x, y are objects the ordered pair of x and y is denoted (x,y)
- in ordered pairs, order matters and repetition is allowed
- e.g. $(1,2) \neq (2,1) \neq (2,2)$

- if A, B are sets then the Cartesian product of A and B is
$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$$

- e.g. $\{1,2\} \times \{*, \heartsuit\}$
$$= \{(1,*), (2,*), (1,\heartsuit), (2,\heartsuit)\}$$

- we write $A \times A$ as A^2

- more generally: ordered n-tuples are denoted $(x_1, x_2, \dots, x_n)$
- sometimes write $\bar{x}$ for $(x_1, \dots, x_n)$

- if $A_1, \dots, A_n$ are sets then their Cartesian product is:

$$A_1 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for every } i \leq n\}$$

- if these are all the same set we write
$$A \times \cdots \times A = A^n$$

- e.g. if $A = \{0,1\}$
then $A^3 = \{0,1\} \times \{0,1\} \times \{0,1\}$
$= \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$

- by convention, for any set A we have $A^0 = \{\emptyset\}$

Relations and Functions

Def'n if A is a set, an *n-ary relation* on A is a subset $R \subseteq A^n$

- e.g. if $A = \{1,2,3\}$ then $R = \{(1,1), (2,3), (2,1)\} \subseteq A^2$ is a 2-ary relation on A
- 2-ary relations also called *binary*

- if R is an n-ary relation on A and $\bar{x} \in A^n$ then $R(\bar{x})$ and $\bar{x} \in R$ mean the same thing

- for binary relations we often write xRy instead of $(x,y) \in R$

- e.g. for relation R above we could write $2R1$ instead of $(2,1) \in R$

- if $\leq$ is the usual "less than or equal to" relation on $\mathbb{R}$ we will write $2 \leq \pi$ instead of $(2, \pi) \in \leq$.

"such that"

<u>Def'n</u> if A, B are sets, a function from A to B is a subset $f \subseteq A \times B$ s.t.

for every $x \in A$ there is a <u>unique</u> $y \in B$ s.t. $(x,y) \in f$.

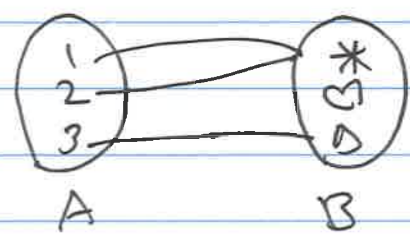
- will usually write $f(x) = y$ to mean $(x,y) \in f$
- will write $f: A \to B$ to mean f is a function from A to B.
- A is called domain of f, B is called codomain of f

- e.g. if $A = \{1, 2, 3\}$ and
  $B = \{*, \heartsuit, \Delta\}$

then

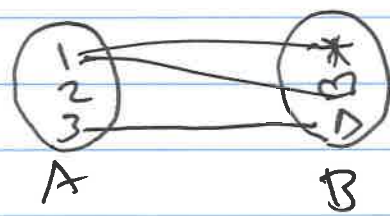
$$f = \{(1, *), (2, *), (3, \Delta)\}$$

is a function from A to B



$f(1) = f(2) = *$
$f(3) = \Delta$

whereas $g = \{(1, *), (1, \heartsuit), (3, \Delta)\}$
is *not*



$g(1)$ not unique
$g(2)$ not defined

- sometimes define functions with rules, e.g. "let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$"
- then really $f = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$

- if $f: A \to B$ is a function and $X \subseteq A$ then $f[X]$ denotes the set $\{f(x) \mid x \in X\}$
- ~~and~~ $f[X]$ is called the image of X

- $f[A]$ is the image of the domain A, also called <u>range of f</u>.

- e.g. if $f: \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$ and $X = \{1,2,3\}$ then

$$f[X] = \{f(1), f(2), f(3)\}$$
$$= \{1, 4, 9\}$$

and
$$f[\mathbb{R}] = \text{range}(f)$$
$$= \{f(x) \mid x \in \mathbb{R}\}$$
$$= \{x^2 \mid x \in \mathbb{R}\}$$
$$= \{x \mid x \geq 0\}$$

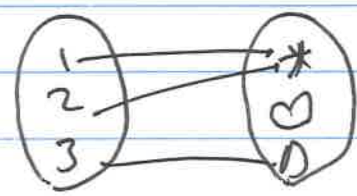
- if $f: A \to B$ is a function and $Y \subseteq B$ then $f^{-1}[Y]$ denotes the set $\{x \in A \mid f(x) \in Y\}$
- $f^{-1}[Y]$ is called the <u>preimage of Y</u>.

- e.g. if $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is our function above and $Y = \{1,4,9\}$ then

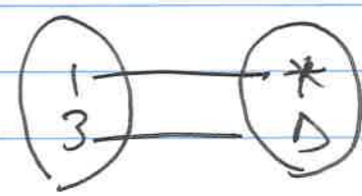
$$f^{-1}[Y] = \{x \in \mathbb{R} \mid f(x) \in Y\}$$
$$= \{x \in \mathbb{R} \mid x^2 \in Y\}$$
$$= \{x \in \mathbb{R} \mid x^2 \in \{1,4,9\}\}$$
$$= \{-3, -2, -1, 1, 2, 3\}$$

- if $f: A \to B$ is a function and $X \subseteq A$ then the <u>restriction of f</u> to X, written $f \restriction X$, is the function
$\{(x,y) \in f \mid x \in X\}$

- e.g. if $f: \{1,2,3\} \to \{*, \heartsuit, \Delta\}$
$f = \{(1,*), (2,*), (3,\Delta)\}$
is our function from before and
$X = \{1,3\}$ then $f \restriction X = \{(1,*), (3,\Delta)\}$



f



$f \restriction X$

- $f \restriction X$ is always a function w/ <u>domain</u> <u>X</u>.

"suppose"

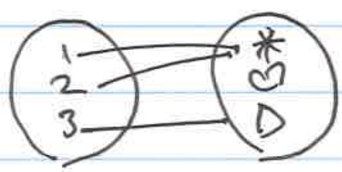
<u>Def'n</u> Sps $f: A \to B$ is a function
- f is an <u>injection</u> (or 1-1) if for every $x, x' \in A$, if $x \neq x'$ then $f(x) \neq f(x')$
("distinct inputs yield distinct outputs")

- f is a <u>surjection</u> (or <u>onto</u>)

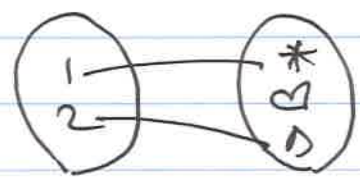
if for every $y \in B$ there is $x \in A$ s.t. $f(x) = y$
("every possible output is attained")

- f is a <u>bijection</u> if f is both an injection and surjection.

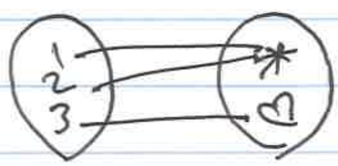
<u>Picture</u>:



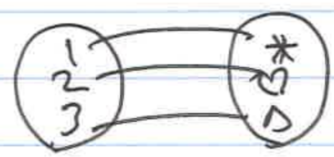
neither injection nor surjection



injection but not surjection



surjection but not injection



bijection

<u>Compositions</u>

- if $f: A \to B$ and $g: B \to C$ are functions then $g \circ f: A \to C$ is a function defined by $g \circ f(a) = g(f(a))$ for every $a \in A$.

- e.g. ~~f~~ $f: \mathbb{Z} \to \mathbb{N}$ is defined by $f(z) = |z|$ and $g: \mathbb{N} \to \mathbb{R}$ is defined by $g(n) = \sqrt{n}$ then

$$g \circ f : \mathbb{Z} \to \mathbb{R}$$

and for every $z \in \mathbb{Z}$ we have

$$g \circ f(z) = g(f(z)) = g(|z|)$$
$$= \sqrt{|z|}$$

- e.g. $g \circ f(-5)$
$$= g(f(-5)) = g(5) = \sqrt{5}$$

Facts: Suppose $f: A \to B$ and $g: B \to C$ are functions.

① if f and g are both injections, then $g \circ f : A \to C$ is an injection

② if f and g are both surjections then $g \circ f : A \to C$ is a surjection

③ if f and g are both bijections, then $g \circ f : A \to C$ is a bijection.

- if A is a set, an n-ary function on A is a function $f: A^n \to A$.
- e.g. $f: \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = x^2 y$ is a binary function on $\mathbb{R}$.

The natural numbers

- $\omega = \mathbb{N} = \{0, 1, 2, 3, \dots\}$
- we'll use set-theoretic convention that each natural number n is a set consisting of previous natural numbers
- so:

$$0 = \emptyset$$
$$1 = \{0\} = \{\emptyset\}$$
$$2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$$
$$3 = \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$
$$\vdots$$

etc.

- Be careful: sometimes we think about 0 "as 0" sometimes "as $\emptyset$".
- If f is a function w/ $\mathrm{dom}(f) = \omega$ we sometimes think of f as an infinite sequence and write
$$f = \langle f(0), f(1), f(2), \dots \rangle$$
- e.g. if $f: \omega \to \omega$ is defined by $f(n) = n^2$

then
$$f = \langle 0, 1, 4, 9, \ldots \rangle$$

- sim'ly: if $\mathrm{dom}(f) = n = \{0, 1, \ldots, n-1\}$
we sometimes write:
$$f = \langle f(0), f(1), \ldots, f(n-1) \rangle$$

- e.g. the sequence
$$\langle 1, 1, \pi, 492 \rangle$$
represents the function $f : 4 \to \mathbb{R}$
- with $f(0) = f(1) = 1$
$f(2) = \pi$
$f(3) = 492$

Induction

Thm Let $P(n)$ be a statement about n, e.g.
"n has a prime factorization"
"$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$"

Suppose
1. $P(0)$ holds
2. for every $n \in \mathbb{N}$,
if $P(n)$ holds, then $P(n+1)$ holds

<u>Then</u> for every $n \in \mathbb{N}$, $P(n)$ holds

<u>Thm</u> (Strong Induction)

Suppose
1. $P(0)$ holds
2. For every $n \in \mathbb{N}$,
   if, for every $k \leq n$, $P(k)$ holds,
   then $P(n+1)$ holds

<u>Then</u> for every $n \in \mathbb{N}$, $P(n)$ holds

<u>Recursion</u>:

<u>Theorem</u> (Recursion Theorem)
- Let B be a set
- Let $P = \{f \mid f$ is a function,
  $\mathrm{dom}(f) \in \omega$
  $\mathrm{ran}(f) \subseteq B\}$
- Suppose $G : P \to B$ is a function.

<u>Then</u> there is a unique function
$F : \omega \to B$ s.t. for every $n \in \omega$
$F(n) = G(F \restriction n)$

- we'll take induction principle + recursion theorem for granted. (i.e. no proofs)
- recursion theorem looks wonky: used to justify recursive def'ns

- e.g. (informal recursive def'n)
define $f: \omega \to \omega$ by

$$f(0) = 0$$
$$f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \quad \text{if } n \geq 2.$$

- so $f(2) = f(0) + f(1) = 0 + 1 = 1$
$f(3) = 1 + 1 = 2$
$f(4) = 1 + 2 = 3$
$f(5) = 2 + 3 = 5$
$f(6) = 3 + 5 = 8$ etc...

- Fibonacci Sequence.

Formal def'n using recursion theorem

let $P = \{f \mid f$ is a function, $\mathrm{dom}(f) \in \omega$, $\mathrm{ran}(f) \subseteq \omega\}$,

"$\langle\ \rangle$"

e.g. $\phi \in P$

$\langle 1, 1, 0, 5 \rangle \in P$

Define $G : P \to \omega$ by:

$G(f) = 0$ if $\mathrm{dom}(f) = 0 = \phi$
(i.e. if $f = \phi$)

$G(f) = 1$ if $\mathrm{dom}(f) = 1$

$G(f) = f(n-2) + f(n-1)$ if $\mathrm{dom}(f) = n$
$n \geq 2$.

So e.g. if $f : 4 \to \omega$ is $\langle 2, 1, 5, 19 \rangle$
then $G(f) = 5 + 19 = f(2) + f(3)$.
$= 24$.

Recursion Thm says exists unique $F : \omega \to \omega$
s.t. for all $n \in \omega$
$F(n) = G(F \upharpoonright n)$

- what does this F look like?

$F(0) = G(F \upharpoonright 0) = G(\langle \rangle)$
$= 0$

$F(1) = G(F \upharpoonright 1) = G(\langle 0 \rangle)$
$= 1$

$F(2) = G(F \upharpoonright 2) = G(\langle 0, 1 \rangle)$
$= 0 + 1 = 1$

$F(3) = G(F \upharpoonright 3) = G(\langle 0, 1, 1 \rangle)$
$= 1 + 1 = 2$

$F(4) = G(F \upharpoonright 4) = G(\langle 0, 1, 1, 2 \rangle)$
$= 1 + 2 = 3$

etc: we see F enumerates the Fibonacci sequence.

$F = \langle 0, 1, 1, 2, 3, 5, 8, \ldots \rangle$

- in particular $F(n) = F(n-2) + F(n-1)$ for all $n \geq 2$.

- General hint: to prove things about recursively defined functions, use induction

ex: For $F: \omega \to \omega$ defined recursively above, prove that for every $n \in \omega$ we have

$$\sum_{k=0}^{n} F(k) = F(n+2) - 1$$

Pf: (BC): if $n = 0$, then

$$\sum_{k=0}^{0} F(k) = F(0) = 0 = 1 - 1 = F(0+2) - 1$$

and the statement holds

(IH) Fix $n \in \omega$.
Assume $\sum_{k=0}^{n} F(k) = F(n+2) - 1$

Then

$$\sum_{k=0}^{n+1} F(k) = \sum_{k=0}^{n} F(k) + F(n+1)$$
$$\overset{IH}{=} F(n+2) - 1 + F(n+1)$$
$$= F(n+3) - 1$$
$$= F((n+1)+2) - 1$$

By induction, statement holds for all $n \in \omega$.

Cardinality

Def'n a set S is finite iff for some $n \in \omega$ there is a surjection $f: n \to A$.

ex. $S = \{*, \heartsuit, \Delta\}$ is finite

Pf: define
$f: S \to S$ by

$f(1) = *$
$f(2) = f(5) = \heartsuit$ then f is
$f(3) = f(4) = \Delta$ a surjection ✓

Def'n A set S is infinite iff it is not finite

ex $\mathbb{Z}$ is infinite

Pf: If $n \in \omega$ and $f: n \to \mathbb{Z}$ then f is not a surjection. (Why?)

Def'n A set S is countable iff there is a surjection $f: \omega \to S$.

Ex's ① ω is ctbl

Pf: $f: \omega \to \omega$ defined by $f(n) = n$ is a bijection, hence surjection

② Fix $n \in \omega$. Then $n = \{0, 1, \ldots, n-1\}$ is ctbl.

Pf: define $f: \omega \to n$ by

$f(k) = k$ if $k < n$

$f(k) = n-1$ if $k \geq n$

so $f = \langle 0, 1, 2, \ldots, n-1, n-1, \ldots \rangle$.

Then f is a surjection ✓

③ Any finite set S is ctbl.

Pf: - Fix $n \in \omega$ s.t. there is a surjection $g: n \to S$

- Let $f: \omega \to n$ be surjection

from ②. Then $g \circ f : \omega \to S$ is a surjection.

④ $\mathbb{Z}$ is ctbl.
<u>Pf</u>: define $f : \omega \to \mathbb{Z}$ by

$$f(n) = \begin{cases} -n/2 & \text{if } n \text{ is even} \\ n+1/2 & \text{if } n \text{ is odd} \end{cases}$$

so $f(0) = 0$ $f(1) = 1$ $f(2) = -1$
$f(3) = 2$ $f(4) = -2$
$f(5) = 3$ $f(6) = -3$

f, so defined, is a bijection (why?)

<u>Fact</u>: Sps A, B are nonempty sets. There is a surjection from A to B iff there is an injection from B to A

<u>Pf</u> $(\Rightarrow)$ - Sps $f : A \to B$ is a surjection.
- For each $b \in B$, choose (AC) an $a_b \in A$ s.t. $f(a_b) = b$ (exists by surjection)
- Define $g : B \to A$ by $g(b) = a_b$
- then g is an injection:
if $b \neq b'$ then cannot be that $a_b = a_{b'}$

- since $f(a_b) = b \neq b' = f(a_{b'})$
- hence $a_b \neq a_{b'}$, i.e. $g(b) \neq g(b')$ ✓

($\Leftarrow$) Sps $g: B \to A$ is an injection. Pick $b_0 \in B$.
Define $f: A \to B$ as follows:
given $a \in A$, if there is a $b \in B$ s.t. $g(b) = a$ let $f(a) = b$. If there is no such b let $f(a) = b_0$.
Then f is a surjection (Why?)

Hence:
<u>Prop'n</u> Sps S is nonempty. If there is an injection $f: S \to \omega$ then S is ctbl.
Pf: use Fact. ✓

<u>Def'n</u> Let A be a set. $A^{<\omega}$ denotes set of all finite tuples of el'ts of A
i.e.

$$A = A^0 \cup A^1 \cup \cdots$$
$$= \bigcup_{n \in \omega} A^n$$

<u>Prop'n</u> $\omega^{<\omega}$ is countable.

We will define an injection
$f: \omega^{<\omega} \to \omega$

- Let $p_1, p_2, p_3, \ldots$ be an increasing enumeration of the primes
- so $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc..

- for $(n_1, \ldots, n_k) \in \omega^{<\omega}$ define
$f((n_1, \ldots, n_k)) = p_1^{n_1+1} p_2^{n_2+1} \cdots p_k^{n_k+1}$

- so e.g. $f((3,1)) = p_1^{3+1} p_2^{2+1} = 2^{3+1} \cdot 3^{2+1}$
$= 2^4 \cdot 3^3$
$= 432$

$f((0,0,1)) = p_1^{0+1} p_2^{0+1} p_3^{1+1}$
$= 2^1 3^1 5^2$
$= 150$

- Claim f is an injection.
why. if $(n_1, \ldots, n_k) \neq (l_1, \ldots, l_m)$ are distinct tuples then
$p_1^{n_1+1} \cdots p_k^{n_k+1} \neq p_1^{l_1+1} \cdots p_m^{l_m+1}$.
by fund. thm. of arithmetic
i.e. $f((n_1, \ldots, n_k)) \neq f((l_1, \ldots, l_m))$ ✓

<u>Prop'n</u> Sps A, B are nonempty sets with $A \subseteq B$. If B is cbl, then A is cbl.

<u>Pf</u>: Fix $a_0 \in A$.
Suppose $f: \omega \to B$ is a surjection
define $g: \omega \to A$ by
$$g(n) = f(n) \quad \text{if } f(n) \in A$$
$$g(n) = a_0 \quad \text{if } f(n) \notin A$$
g is a surjection ✓ (why?)

<u>ex</u>. ① Let $E = \{0, 2, 4, \dots\}$
then $E \subseteq \omega$ so E is cbl ✓
② For any fixed n, $\omega^n \subseteq \omega^{<\omega}$.
Hence ω^n is cbl.
In particular $\omega \times \omega = \omega^2$ is cbl.
③ $\mathbb{Q}$ is cbl.
<u>Pf</u>: there is a surjection
$f: \omega \times \omega \to \mathbb{Q}$ (why?)

<u>Uncountable sets</u>

Not all infinite sets are countable

<u>Def'n</u>: A set S is uncountable if S is not countable, i.e. if there is no surjection $f: \omega \to S$.

– Recall: $P(\omega) = \{x \mid x \subseteq \omega\}$

<u>Claim</u> $P(\omega)$ is uncountable.

<u>Pf</u>: Fix a function $f : \omega \to P(\omega)$

We prove f is not a surjection

Define a subset $T \subseteq \omega$

$T = \{n \in \omega \mid n \notin f(n)\}$

We argue that for every $n \in \omega$, $f(n) \neq T$

<u>Case 1</u>: $n \in f(n)$.

Then $n \notin T$. Hence $f(n) \neq T$

<u>Case 2</u>: $n \notin f(n)$

Then $n \in T$. Hence $f(n) \neq T$.

Hence, as claimed, $f(n) \neq T$ for every $n \in \omega$.

Hence f not a surjection

↳ no surjection $f : \omega \to P(\omega)$ exists.

<u>Def'n</u> 2^ω denotes $\{f \mid f : \omega \to 2\}$

↳ think of el'ts of 2^ω as infinite 0,1-sequences

e.g. $\langle 0, 1, 1, 0, 0, 0, \ldots \rangle \in 2^\omega$

Prop'n 2^ω is unctbl

Pf. For every $A \in P(\omega)$ define $f_A : \omega \to 2$ by

$$f_A(n) = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if } n \notin A \end{cases}$$

e.g. if $A = \{0, 2, 4, \ldots\}$ then $f_A = \langle 1, 0, 1, 0, \ldots \rangle$

– the map $F : P(\omega) \to 2^\omega$ defined by $F(A) = f_A$ is a bijection (why?)

– hence 2^ω is unctbl (why?) ✓

Fact $\mathbb{R}$ is unctbl (why?)

Equivalence Relations

Def'n Sps S is a set and $E \subseteq S \times S$ is a binary relation on S.

- E is reflexive if for all $x \in S$, $(x,x) \in E$
- E is symmetric if for all $x, y \in S$ if $(x,y) \in E$ then $(y,x) \in E$
- E is transitive if for all $x, y, z \in S$ if (x,y) and $(y,z) \in S$ then $(x,z) \in E$

- E is an equivalence relation if E is refl., sym., and transitive.

ex. Define a relation $E \subseteq \mathbb{Z} \times \mathbb{Z}$ by $(x,y) \in E$ if $x^2 = y^2$

$\hookrightarrow$ e.g. $(-2, 2) \in E$.

Claim This E is an equiv relation.

Pf Fix $x, y, z \in \mathbb{Z}$.

1. Observe $x^2 = x^2$ hence $(x,x) \in E$
2. If $(x,y) \in E$ then $x^2 = y^2$, hence $y^2 = x^2$, hence $(y,x) \in E$
3. If $(x,y), (y,z) \in E$ then $x^2 = y^2$ and $y^2 = z^2$, hence $x^2 = z^2$, hence $(x,z) \in E$

Since x, y, z were arbitrary E is equiv relation. ✓

Nonex. $\leq$ is not an equiv. relation on $\mathbb{R}$.

why: $\leq$ is reflexive and transitive but not symmetric, e.g. $3 \leq 5$ but $5 \not\leq 3$.

Def'n if E is an equivalence relation on S and $x \in S$, the equivalence class of x is

$$[x]_E = \{ y \in S \mid (x,y) \in E \}$$

= "set of things equiv. to x"

e.g. in our example above

$$[3]_E = \{ y \in \mathbb{Z} \mid (3,y) \in E \}$$
$$= \{ y \in \mathbb{Z} \mid 3^2 = y^2 \}$$
$$= \{-3, 3\}$$

sim'ly for any $z \in \mathbb{Z}$
$[z]_E = \{z, -z\}$.

↳ if E is an equiv. relation on S, the set of equivalence classes is denoted S/E

i.e.

$$S/E = \{ [x]_E \mid x \in S \}$$

ex: in above example we have

$$\mathbb{Z}/E = \{ [x]_E \mid x \in \mathbb{Z} \}$$
$$= \{ \ldots, [-1]_E, [0]_E, [1]_E, [2]_E, \ldots \}$$
$$= \{ \{0\}, \{-1,1\}, \{-2,2\}, \ldots \}$$

Def'n if S is a set and P is a collection of subsets of S (i.e. $P \subseteq \mathcal{P}(S)$) then P is a partition of S if

① for every $X \in P$, $X \neq \emptyset$
② for every $X, Y \in P$, either $X \cap Y = \emptyset$ or $X = Y$ (nonoverlap)
③ $\bigcup P = S$.

ex: let $P = \{ \{0\}, \{-1,1\}, \{-2,2\}, \ldots \}$

Then P is a partition of $\mathbb{Z}$

<u>why</u>: ① all sets in P nonempty ✓

② sets in P don't overlap (are pairwise disjoint) ✓

③ $\cup P = \mathbb{Z}$ ✓

<u>nonex</u>: let $E = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$

$O = \{\ldots, -1, 1, 3, 5, \ldots\}$

$\omega = \{0, 1, 2, \ldots\}$

Then $P = \{E, O, \omega\}$ is <u>not</u> a partition of $\mathbb{Z}$.

② fails: $E \cap \omega \neq \emptyset$.

<u>ex</u>: let $A = \{x \in \mathbb{R} \mid x \geq 0\}$

$B = \{x \in \mathbb{R} \mid x < 0\}$

then $P = \{A, B\}$ is a partition of $\mathbb{R}$ (why?)

Now: define a relation $E \subseteq \mathbb{R}^2$ by $(x,y) \in E$ if x, y lie in same piece of partition P above

so e.g. $(1, \pi) \in E$

$(-3, -7) \in E$

but $(-3, \pi) \notin E$

Claim E is an equiv relation on $\mathbb{R}$

Pf ① x always has same sign as itself $\Rightarrow (x,x) \in E$

② if x,y have same sign then y,x have same sign i.e. $(x,y) \in E \Rightarrow (y,x) \in E$

③ if x,y have same sign and y,z have same sign then x,z have same sign ~~then~~ i.e. $(x,y) \in E$ and $(y,z) \in E \Rightarrow (x,z) \in E$ ✓

Will prove on HW this example not an accident.