Ex: Sps we roll n (indistinguishable) 6-sided dice.

How many distinct outcomes are possible?

Sol'n: each of the n dice can roll into 6 possible "types"

1	2	3	4	5	6
oo	o		ooo	o	oo

hence # of possible outcomes is:

$$\binom{n+(6-1)}{6-1} = \binom{n+5}{5}$$

So if we roll 10 dice, # is:

$$\binom{15}{5} = 3003.$$

Ex: How many anagrams of the word

LIMITING

are there?

Sol'n: Two approaches:

① first distinguish the I's w/ subscripts I_1, I_2, I_3

- # of anagrams w/ distinguished I's is just $8!$
- For each anagram w/ distinguished I's there are $3!$ equivalent anagrams (including itself) w/ I's not distinguished.
~~[illegible]~~ (one for each permutation of $I_1 I_2 I_3$)
- Hence: # of anagrams is: $\frac{8!}{3!} = 6720$

② Can think of anagram being formed in two stages:

(i) Pick 3 positions for the I's $\binom{8}{3}$

(ii) For remaining 5 positions pick ordering of LMTNG : $5!$

$\Rightarrow$ # of anagrams $= \binom{8}{3} 5!$

$$= \frac{8!}{3!5!} 5! = \frac{8}{3!}$$

as before!

Countin' in Two Ways:

Thm (Pascal's identity) Fix $n, k \in \mathbb{N}$ with $k \leq n$. Then:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Pf. Let S be the set of k element subsets of $[n] = \{1, 2, \ldots, n\}$

Then $|S| = \binom{n}{k}$

OTOH: we can partition S into S_1 and T where: S_1 = k element subsets of $[n]$ that contain 1

T = k el't subsets of $[n]$ that don't contain 1

Then: $|S| = |S_1| + |T|$

observe: subsets in S_1 are formed by selecting $k-1$ elements from $\{2, 3, 4, \ldots\}$ (1 is auto included) $\Rightarrow |S_1| = \binom{n-1}{k-1}$

subsets in T are formed by selecting k elements from $\{2, 3, \ldots, n\} \Rightarrow |T| = \binom{n-1}{k}$

Hence: $|S| = \binom{n-1}{k-1} + \binom{n-1}{k}$

i.e. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ ✓

Claim: Fix $n, k \in \mathbb{N}$, $k \leq n$. Then:

$$\binom{n}{k} k = n \binom{n-1}{k-1}$$

Pf: Let S denote the set of committees of k ppl chosen from a group of n ppl, w/ a specified chairperson.

Such a committee can be formed by:

- picking the committee members $\binom{n}{k}$
- from these, picking the chair $\binom{k}{1} = k$.

$\Rightarrow |S| = \binom{n}{k} k$

Or can form a committee by: (2c)

- picking a chair first: $\binom{n}{1} = n$
- from remaining $n-1$ ppl, choose remaining $k-1$ committee members $\binom{n-1}{k-1}$

hence $|S| = n\binom{n-1}{k-1}$ too

$\Rightarrow n\binom{n-1}{k-1} = \binom{n}{k}k$ ✓

---

in this case: can verify identity algebraically: $\binom{n}{k}k = \frac{n!}{k!(n-k)!}k$

$$= \frac{n!}{(k-1)!(n-k)!}$$

$$n\binom{n-1}{k-1} = n\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!}$$

$$= \frac{n!}{(k-1)!(n-k)!}$$

same

Prop'n: Fix $n \in \mathbb{N}$. Then:

$$n 2^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k$$

Pf: Let S be the set of nonempty committees w/ a chairperson chosen from a group of n people.

Done ✓ Why:

Can form a committee by:

- choosing the chair $\binom{n}{1} = n$
- from remaining $n-1$ ppl, choosing other committee members (i.e. just choose a subset from a set of size $n-1$) 2^{n-1}

$\Rightarrow |S| = n 2^{n-1}$

OTOH we can partition S:

$S = A_1 \cup A_2 \cup \cdots \cup A_n$

where A_K is the set of committees w/ exactly K people.

above we computed

$$|A_K| = \binom{n}{K} K.$$

Hence $|S| = |A_1| + |A_2| + \cdots + |A_n|$

$$= \binom{n}{1} 1 + \binom{n}{2} 2 + \cdots + \binom{n}{n} n$$

$$= \sum_{K=1}^{n} \binom{n}{K} K \checkmark$$

As a bonus, using previous example, could ~~rew~~ write this identity eq:

$$n 2^{n-1} = \sum_{K=1}^{n} \binom{n}{K} K = \sum_{K=1}^{n} n \binom{n-1}{K-1} \checkmark$$

Inclusion / Exclusion :

ROS says: If we write a set A

as:

$$A = A_1 \cup A_2 \cup \cdots \cup A_K$$

and then A_i's are disjoint, then

$|A| = |A_1| + |A_2| + \cdots + |A_k|$.

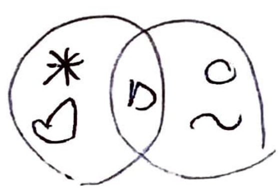
... but what if the A_i's are not disjoint? Can we still count $|A|$ in terms of the $|A_i|$'s?

__Yes__: but need the principle of inclusion/exclusion.

__ex__: Let $A_1 = \{*, \heartsuit, \Delta\}$
$A_2 = \{\Delta, \circ, \sim\}$

Let $A = A_1 \cup A_2$

What is $|A|$?



A_1 A_2

is $|A| = |A_1| + |A_2|$? Not quite — since $\Delta \in A_1 \cap A_2$.

but we can think of counting $|A|$ as $|A_1| + |A_2|$ ... then correcting over-counting:

In this case we count the elements in $A_1 \cap A_2$ (i.e. $\triangle$) twice, hence

$$
\begin{aligned}
|A| &= |A_1| + |A_2| - |A_1 \cap A_2| \\
&= 3 + 3 - 1 \\
&= 5 \checkmark
\end{aligned}
$$

and indeed $A = A_1 \cup A_2 = \{*, \heartsuit, \triangle, \circ, \sim\}$

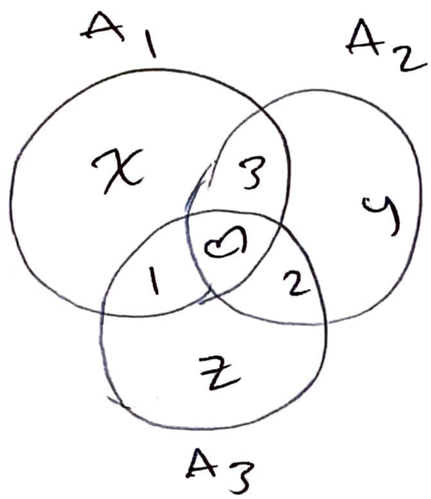
→ this is general!

For any finite set A, if $A = A_1 \cup A_2$ then $|A| = |A_1| + |A_2| - |A_1 \cap A_2|$

what about 3 sets?

<u>ex</u>: sps

$$
\begin{aligned}
A_1 &= \{x, 1, 3, \heartsuit\} \\
A_2 &= \{y, 2, 3, \heartsuit\} \\
A_3 &= \{z, 1, 3, \heartsuit\}
\end{aligned}
$$



$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$$

$$- |A_1 \cap A_2| - |A_1 \cap A_3|$$

el'ts here counted twice

$$- |A_2 \cap A_3|$$

$$+ |A_1 \cap A_2 \cap A_3|$$

el'ts here counted twice, <u>then</u> subtracted three times!

so need to add back!