

Selections: Def'n: Fix $n, k \in \mathbb{N} \setminus \{0\}$ ⑦
with $k \leq n$. Given A , with $|A| = n$, a
 k -selection (or k -combination) is a subset
of A of size k (i.e. an unordered list
of el's of A)

e.g. $\{2, 3\}$ is a 2-selection of $\{1, 2, 3, 4\}$.

Notation $\binom{n}{k}$ denotes # of k -selections
from a set of size n .

Prop'n Fix n, k with $k \leq n$.

Then:
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

PF: - Spcs A is a set of size n , and
let S be the set of k -arrangements
of A .

- We count $|S|$ in two ways.

- We already know:

$$|S| = \frac{n!}{(n-k)!}$$

CTOH: an element of S (i.e. a k -arrangement of A) can be formed in two steps:

First choose a k -selection (a_1, a_2, \dots, a_k) from A

then check an ordering (i.e. permutation) of this selection.

- there are (by def'n) $\binom{n}{k}$ many ways to make first choice, and (by ROP) $k!$ ways to make second.

- Hence $|S| = \binom{n}{k} k!$ (by ROP)

- Hence $\binom{n}{k} k! = \frac{n!}{(n-k)!}$

$$\Rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \checkmark$$

e.g. $\binom{10}{3} = \frac{10!}{7!3!} = \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!} \cdot 3 \cdot 2 \cdot 1}$

notice $\binom{10}{7} = \frac{10!}{3!7!} = 120$ too

in general: ~~$\binom{n}{k}$~~ $\binom{n}{k} = \binom{n}{n-k}$

Ex: How many ways are there to choose a chief and two benchpersons from a group of 10 ppl? (9)

Sol'n: - 10 choices for chief
- once a chief chosen, $\binom{9}{2}$ choices

For benchppl

\Rightarrow # of such committees is

$$10 \binom{9}{2} = 10 \cdot \frac{9!}{2!}$$

$$= 10 \cdot \frac{9 \cdot 8}{2 \cdot 1}$$

$$= 360$$

Alt. solution: - first select group of 3
then from these ~~3~~ select a
chief

\Rightarrow # of committees possible

$$= \binom{10}{3} \binom{3}{1} = 120 \cdot 3$$

$$= 360, \text{ as before!}$$

Counting Poker Hands:

①

- a standard deck consists of 52 cards
- each card has 1 of 4 suits
(♠, ♡, ♣, ♣)

and 1 of 13 ranks:

(A, 2, 3, ..., 9, 10, J, Q, K)

- e.g. A ♠ and ~~Q♠~~ 9♠ are cards
- a poker hand is a 5-selection from a standard deck.

Ex's ① How many distinct hands are possible?

Sol'n: $\binom{52}{5} = 2,598,960.$

② A full house is a hand consisting of 3 cards of one rank and 2 cards of another, e.g. A ♠, A ♠, 3 ♠, 3 ♠, 3 ♠

↳ How many distinct full house hands are possible?

Sol'n: - pick two ranks $\binom{13}{2}$ (11)
 - from these, pick the 3-card rank $\binom{2}{1}$
 - pick three cards from this rank $\binom{4}{3}$
 - pick two cards from the 2-card rank $\binom{4}{2}$

\Rightarrow # of Full house hands is:

$$\binom{13}{2} \binom{2}{1} \binom{4}{3} \binom{4}{2} = 3,744$$

③ A 3-of-a-kind consists of 3 cards from a single rank and 2 other cards from two other distinct ranks, e.g. 3 Q's, a 10, and a K.

Q: How many 3-of-a-kind hands are possible?

Sol'n: - pick 3-card rank $\binom{13}{1}$
 - from this rank, pick 3 cards $\binom{4}{3}$

- pick remaining two ranks $\binom{12}{2}$ $\textcircled{12}$
- from the first, pick a card $\binom{4}{1}$
- also from the second $\binom{4}{1}$

\Rightarrow # of 3-of-a-kind hands

$$= \binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{1} \binom{4}{1} = 54,912.$$

Alt Sol'n: - Pick three ranks $\binom{13}{3}$

- from them, pick 3-card rank $\binom{3}{1}$

- pick three cards from this rank $\binom{4}{3}$

- from other two ranks, pick cards $\binom{4}{1} \binom{4}{1}$.

$$\text{But: } \binom{13}{3} \binom{3}{1} \binom{4}{3} \binom{4}{1} \binom{4}{1} = 54,912$$

too ✓

Binary sequences: a binary sequence (of length n) is an ordered sequence of 0's and 1's (of length n)

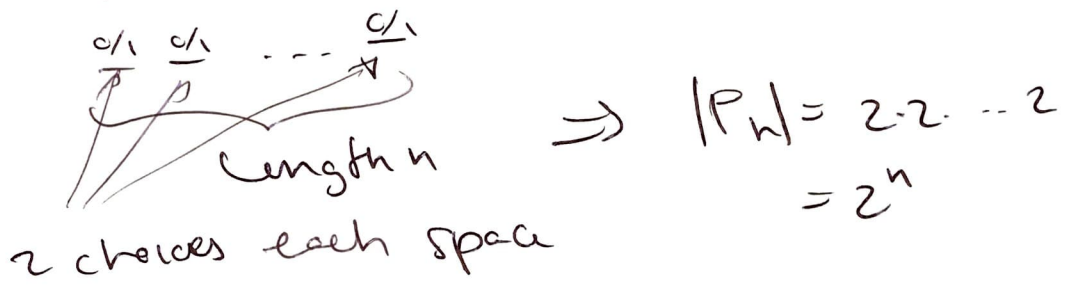
e.g. 011 and 101 are binary sequences of length 3.

- let P_n denote the set of binary sequences of length n .

Ex: ① What is $|P_n|$?

② How many sequences in P_n have at least two 1's?

Soln: ① each in P_n formed by making a sequence of n choices:



② easier to count # of seq's w/ either zero 1's or one 1, then subtract.

w/ zero 1's = 1 (just 1...1)

w/ one 1 = n (one for each place to put the 1)

$$\Rightarrow \# \text{ w/ at least two 1's} \\ = 2^n - n - 1$$

Theorem: Fix $n \in \mathbb{N} \cup \{0\}$.

$$\text{then } 2^n = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} \\ = \sum_{k=0}^n \binom{n}{k}$$

Pf: if $n=0$, then $2^0 = 1 = \binom{n}{0} = \sum_0^0 \binom{n}{k}$

So sps $n \geq 1$.

We proved $|P_n| = 2^n$

We can partition $P_n = S_0 \cup S_1 \cup \dots \cup S_n$

where $S_k =$ set of sequences w/ exactly k -many 1's.

observe: $S_k = \binom{n}{k}$ \leftarrow # of ways to pick k positions where 1's appear

Hence: $2^n = |P_n| = |S_0| + |S_1| + \dots + |S_n| \\ = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} \\ = \sum_{k=0}^n \binom{n}{k} \checkmark$

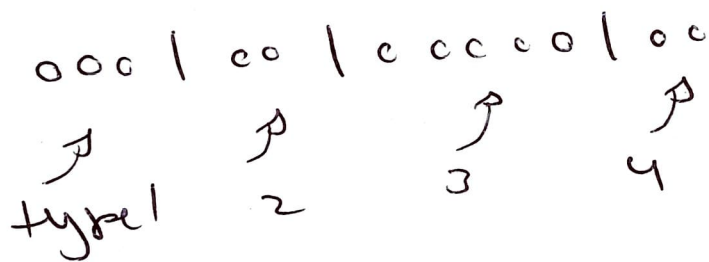
Selections w/ repetition:

(K)

Q: How many ways are there to select n objects from k types of objects, if repetition allowed?

Ex's: Dee's donuts sells 4 types of donuts, and you want to buy a dozen. How many distinct ways to do this?

Sol'n: Imagine putting down 3 "spaces" to separate donut types:



above "donut + spacer" diagram would correspond to purchase of:

3 donuts of type 1
2 of type 2
5 of type 3
2 of type 4

→ can view the diagram as a CI-sequence
w/ 12 o's (for donuts) 3 l's (to separate 4 types)

→ Conversely: any such sequence (12 0's, 3 1's) (16)
corresponds to a selection of 12 donuts:

→ e.g.

1 0 1 0 0 0 0 0 0 0 1 0 0 0 0

corresponds to:

0 tp 1 donuts
1 tp 2 donuts
7 tp 3 donuts
4 tp 4 donuts.

→ Hence: # of ways to make a selection
= # of 01-sequ w/ 12 0's and 3 1's
= # of 01-sequ of length 15 w/ 3 1's
= $\binom{15}{3} = 455$.

Same reasoning in general proves:

Thm The # of ways to select n objects
from k types w/ repetition allowed is:
$$\binom{n+(k-1)}{k-1}$$

($k-1$ because we need $k-1$ "spaces"
to separate k types of objects).