<u>Selections</u>: <u>Def'n</u>: Fix $n, k \in \mathbb{N} \cup \{0\}$ with $k \leq n$. Given A, with $|A| = n$, a <u>k-selection</u> (or <u>k-combination</u>) is a subset of A of size k (i.e. an <u>unordered</u> <u>list</u> of el'ts of A)

e.g. $\{2,3\}$ is a 2-selection of $\{1,2,3,4\}$.

<u>Notation</u> $\binom{n}{k}$ denotes # of k-selections from a set of size n.

<u>Prop'n</u> Fix n, k with $k \leq n$.

<u>Then</u>: $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

<u>Pf</u>: - Sps A is a set of size n, and let S be the set of k-<u>arrangements</u> of A.

- We count $|S|$ in <u>two</u> <u>ways</u>.
- We already know:

$$|S| = \frac{n!}{(n-k)!}$$

OTOH: an element of S (i.e. a k-arrangement of A) can be formed in two steps:

First choose a k-selection $\{a_1, a_2, \ldots, a_k\}$ from A

then choose an ordering (i.e. permutation) of this selection.

– there are (by def'n) $\binom{n}{k}$ many ways to make first choice, and (by ROP) $k!$ ways to make second.

– Hence $|S| = \binom{n}{k} k!$ (by ROP)

– Hence $\binom{n}{k} k! = \frac{n!}{(n-k)!}$

$$\Rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!} \quad ✓$$

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e.g. $\binom{10}{3} = \frac{10!}{7!\,3!} = \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}\; 3 \cdot 2 \cdot 1}$

$= 120$

notice $\binom{10}{7} = \frac{10!}{3!\,7!} = 120$ too

in general: $\binom{n}{k} = \binom{n}{n-k}$

Ex: How many ways are there to choose a chief and two henchpersons from a group of 10 ppl?

Sol'n: - 10 choices for chief

- once chief chosen, $\binom{9}{2}$ choices for henchppl

$\Rightarrow$ # of such committees is

$$10\binom{9}{2} = 10 \cdot \frac{9!}{7!\,2!}$$

$$= 10 \cdot \frac{9 \cdot 8}{2 \cdot 1}$$

$$= 360$$

Alt. solution: - first select group of 3 then from these select a chief

$\Rightarrow$ # of committees possible is $\binom{10}{3}\binom{3}{1} = 120 \cdot 3$

$= 360$, as before!

Countin' Poker Hands:

- a standard deck consists of 52 cards
- each card has 1 of 4 suits ($\heartsuit, \diamondsuit, \clubsuit, \spadesuit$)

  and 1 of 13 ranks:

  ($A, 2, 3, \ldots, 9, 10, J, Q, K$)

- e.g. $A\diamondsuit$ and $9\heartsuit$ are cards
- a poker hand is a 5-selection from a standard deck.

Ex's ① How many distinct hands are possible?

Sol'n: $\binom{52}{5} = 2{,}598{,}960.$

② A full house is a hand consisting of 3 cards of one rank and 2 cards of another, e.g. $A\diamondsuit, A\heartsuit, 3\spadesuit, 3\diamondsuit, 3\clubsuit$

→ How many distinct full house hands are possible?

Sol'n: - pick two ranks $\binom{13}{2}$

- from these, pick the 3-card rank $\binom{2}{1}$

- pick three cards from this rank $\binom{4}{3}$

- pick two cards from the 2-card rank $\binom{4}{2}$

$\Rightarrow$ # of full house hands is:

$$\binom{13}{2}\binom{2}{1}\binom{4}{3}\binom{4}{2} = 3{,}744$$

3) A <u>3-of-a-kind</u> consists of 3 cards from a single rank and 2 other cards from two other distinct ranks, e.g. 3 Q's, a 10, and a K.

Q: How many 3-of-a-kind hands are possible?

<u>Sol'n</u>: - pick 3-card rank $\binom{13}{1}$

- from this rank, pick 3 cards $\binom{4}{3}$

- pick remaining two ranks $\binom{12}{2}$ (12)
- from the first, pick a card $\binom{4}{1}$
- also from the second $\binom{4}{1}$

$\Rightarrow$ # of 3-of-a-kind hands

$$= \binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1} = 54{,}912.$$

<u>Alt</u> <u>Sol'n</u>: - Pick three ranks $\binom{13}{3}$

- from these, pick 3-card rank $\binom{3}{1}$
- pick three cards from this rank $\binom{4}{3}$
- from other two ranks, pick cards $\binom{4}{1}\binom{4}{1}$.

But: $\binom{13}{3}\binom{3}{1}\binom{4}{3}\binom{4}{1}\binom{4}{1} = 54{,}912$

too ✓

<u>Binary</u> <u>sequences</u>: a <u>binary sequence</u> (of length n) is an ordered sequence of 0's and 1's (of length n)

e.g. 011 and 101 are binary sequences of length 3.

- Let P_n denote the set of binary sequences of length n.

<u>Ex</u>: ① What is $|P_n|$?

② How many sequences $s \in P_n$ have <u>at least two</u> 1's?

<u>Sol'n</u>: ① each $s \in P_n$ formed by making a sequence of n choices:

$\underline{0/1}\ \underline{0/1}\ \cdots\ \underline{0/1}$

length n

2 choices each space

$\Rightarrow |P_n| = 2 \cdot 2 \cdots 2$
$= 2^n$

② easier to count # of seq's w/ either zero 1's or one 1, then subtract.

w/ zero 1's $= 1$ (just $(1 \cdots 1)$)

w/ one 1 $= n$ (one for each place to put the 1)

$\Rightarrow$ # w/ at least two 1's
$= 2^n - n - 1$

<u>Theorem</u>: Fix $n \in \mathbb{N} \cup \{0\}$.
then $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$
$= \sum_{k=0}^{n} \binom{n}{k}$

<u>Pf</u>: if $n=0$, then $2^0 = 1 = \binom{n}{0} = \sum_0^0 \binom{n}{k}$.
So sps $n \geq 1$.
We proved $|P_n| = 2^n$
We can <u>partition</u> $P_n = S_0 \cup S_1 \cup \cdots \cup S_n$
where S_k = set of sequences w/ exactly k-many 1's.
<u>observe</u>: $S_k = \binom{n}{k}$ ← # of ways to pick k positions where 1's appear

<u>Hence</u>: $2^n = |P_n| = |S_0| + |S_1| + \cdots + |S_n|$
$= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$
$= \sum_{k=0}^{n} \binom{n}{k}$ ✓

Selections w/ repetition:

Q: How many ways are there to select n objects from k types of objects, if repetition allowed?

Ex's: Dee's donuts sells 4 types of donuts, and you want to buy a dozen. How many distinct ways to do this?

Sol'n: Imagine putting down 3 "spacers" to separate donut types:

ooo | oo | ooooo | oo

type 1   2   3   4

above "donut + spacer" diagram would correspond to purchase of:

3 donuts of type 1
2 of type 2
5 of type 3
2 of type 4

→ can view the diagram as a 01-sequence w/ 12 0's (for donuts) 3 1's (to separate 4 types)

→ Conversely: any such sequence (12 0's, 3 1's) (16)
corresponds to a selection of 12 donuts:

→ e.g.

1 0 1 0 0 0 0 0 0 0 1 0 0 0 0

corresponds to:

0 tp 1 donuts
1 tp 2 donuts
7 tp 3 donuts
4 tp 4 donuts.

→ Hence: # of ways to make a selection

= # of 01-seqs w/ 12 0's and 3 1's
= # of 01-seqs of length 15 w/ 3 1's
= $\binom{15}{3} = 455$.

Same reasoning in general proves:

<u>Thm</u> The # of ways to select n objects from k types w/ repetition allowed is:

$$\binom{n+(k-1)}{k-1}$$

(k-1 because we need k-1 "spacers" to separate k types of objects).