↳ thm says: $\mathbb{N}$ is as small as possible (14)
for an infinite set: even sets that appear smaller (e.g. E, O, or other infinite subsets of $\mathbb{N}$) are actually not.

↳ OTOH: many sets which appear larger than $\mathbb{N}$ (e.g. $\mathbb{Z}$, $\mathbb{Q}$) are actually the same size.

Def'n: A set X is countable iff $\mathbb{N} \sim X$

Ex's ① $\mathbb{Z}$ is countable.

pf: we already showed

$$f: \mathbb{Z} \to \mathbb{N} \quad f(n) = \begin{cases} 2n & n > 0 \\ 2(-n)+1 & n \leq 0 \end{cases}$$

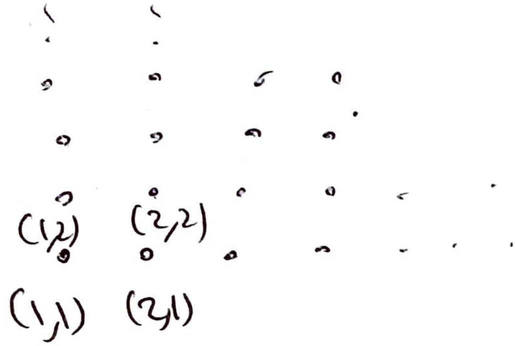
is a bijection

② $\mathbb{N} \times \mathbb{N}$ is countable

pf: need to construct a bijection

$f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$

↳ possible to do this explicitly (i.e. define f w/ a fml(e)), but we just draw a picture.



$\mathbb{N}\times\mathbb{N}$

to construct $f: \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ we "count $\mathbb{N}\times\mathbb{N}$ along its diagonals"

i.e. $f(1) = (1,1)$
$f(2) = (1,2)$
$f(3) = (2,1)$
$f(4) = (1,3)$ etc.

$\vdots$

resulting $f: \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ is injective + surjective

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<u>Theorem</u> (Cantor - Schroeder - Bernstein)
For any sets A, B if $A \lesssim B$ and $B \lesssim A$ then $A \sim B$.

<u>Pf</u>: interesting but tricky — we'll skip and take theorem for granted.

The point: sometimes it's hard to show $A \sim B$ directly (i.e. build a bijection) but ez'er to show $A \lesssim B$ <u>and</u> $B \lesssim A$
<u>thm says</u>: this is enough to guarantee $A \sim B$!

Note: CSB says $A \lesssim B \wedge B \lesssim A \Rightarrow A \sim B$

Since we know: $A \lesssim B$ iff $B \gtrsim A$, CSB also gives that: $B \gtrsim A \wedge A \gtrsim B \Rightarrow A \sim B$.

i.e. $\lesssim$ and $\gtrsim$ are "antisymmetric up to $\sim$"

Next goal: prove that $\mathbb{N} \sim \mathbb{Q}$!

First need: Theorem: if A, B are countable sets then $A \times B$ is ctbl.

Pf: Spr A, B are ctbl, i.e. we have bijections

$$f : \mathbb{N} \to A$$
$$g : \mathbb{N} \to B.$$

We know: $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$

So if we can show: $\mathbb{N} \times \mathbb{N} \sim A \times B$ we'll be done (by transitivity of $\sim$).

Consider $F : \mathbb{N} \times \mathbb{N} \to A \times B$

defined by $F(n, m) = (f(n), g(m))$

<u>Claim</u> F is a bijection.

<u>Pf</u> (surj) – Fix $(a,b) \in A \times B$.

– since f, g both surj., $\exists n, m \in \mathbb{N}$

$f(n) = a \quad f(m) = b$

– hence $F(n,m) = (a,b)$ ✓

(inj.) – if $F(n,m) = F(n',m')$

then $(f(n), g(m)) = (f(n'), g(m'))$

– i.e. $f(n) = f(n')$ and $g(m) = g(m')$

– since f, g both injective this

implies $n = n'$ and $m = m'$

– i.e. $(n,m) = (n',m')$. ✓

hence $\mathbb{N} \times \mathbb{N} \sim A \times B$

hence $\mathbb{N} \sim A \times B$, as desired.

<u>Theorem</u>: $\mathbb{Q}$ is ctbl, i.e. $\mathbb{N} \sim \mathbb{Q}$.

<u>Pf</u>: we'll prove: $\mathbb{N} \overset{①}{\lesssim} \mathbb{Q} \overset{②}{\lesssim} \mathbb{Z} \times \mathbb{N} \overset{③}{\lesssim} \mathbb{N}$

By transitivity this gives $\mathbb{N} \lesssim \mathbb{Q} \wedge \mathbb{Q} \lesssim \mathbb{N}$

which by CSB gives $\mathbb{N} \sim \mathbb{Q}$.

① holds since $\mathbb{Q}$ is infinite, by our previous theorem.

③ holds by the preceding theorem:

We know $\mathbb{Z} \sim \mathbb{N}$ and $\mathbb{N} \sim \mathbb{N}$, hence $\mathbb{Z} \times \mathbb{N} \sim \mathbb{N}$.

So it remains to prove ②. We prove $\gtrsim$ version.

<u>Claim</u>: $\mathbb{Z} \times \mathbb{N} \gtrsim \mathbb{Q}$.

<u>Pf</u>: $F: \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ defined by $F(m,n) = \frac{m}{n}$ is a surjection

(why: given $q \in \mathbb{Q}$, if $q = \frac{m}{n}$ then $F(m,n) = q$!)

<u>Note</u>: F is <u>not</u> injective: e.g. $F(1,2) = F(2,4) = \cdots$

Who cares! We've still shown $\mathbb{Z} \times \mathbb{N} \gtrsim \mathbb{Q}$. ✓

Hence $\mathbb{Q} \lesssim \mathbb{Z} \times \mathbb{N}$.

Combined w/ above, this gives $\mathbb{Q} \sim \mathbb{N}$.

<u>Discussion</u>: - among infinite sets X, $\mathbb{N}$ is "small" in the sense that $\mathbb{N} \lesssim X$ always.

- yet, $\mathbb{N}$ is "large" in that for many sets X which appear larger than $\mathbb{N}$ (e.g. $\mathbb{Z}$, $\mathbb{Q}$) we actually have $X \sim \mathbb{N}$.

Q: Are there infinite sets X for which $\mathbb{N} \nsim X$?

Yes!

<u>Theorem</u> (Cantor): $\mathbb{N} < P(\mathbb{N})$

That is: $\mathbb{N} \lesssim P(\mathbb{N})$ but $\mathbb{N} \not\approx P(\mathbb{N})$

<u>Pf</u>: we know $\mathbb{N} \lesssim P(\mathbb{N})$ since $P(\mathbb{N})$ is infinite.

So need to show $\mathbb{N} \not\approx P(\mathbb{N})$

<u>Claim</u>: Suppose $f: \mathbb{N} \to P(\mathbb{N})$ is a fixed function (<u>any</u> function). Then f is <u>not surjective</u>.

Pf: a magic trick.

let $T = \{n \in \mathbb{N} \mid n \notin f(n)\}$

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to illustrate def'n: e.g. if

$$f(1) = \{1, 7, 10\}$$
$$f(2) = \{1, 3, 5, 7, \ldots\}$$
$$f(3) = \emptyset$$
$$f(4) = \{2, 4, 6, 8, \ldots\}$$

<u>then</u> $1 \notin T$ since $1 \in f(1)$
$2 \in T$ since $2 \notin f(2)$
$3 \in T$ since $3 \notin f(3)$
$4 \notin T$ since $4 \in f(4)$ etc.

so $T = \{2, 3, \ldots\}$ in this case.

Then: $(\forall n \in \mathbb{N})\ f(n) \neq T$

Pf: Fix $n \in \mathbb{N}$.

(i) if $n \in T$, then $n \notin f(n)$, by def'n of T. Hence $f(n) \neq T$, since $n \in T$, $n \notin f(n)$

(ii) if $n \notin T$, then $n \in f(n)$, by def'n of T. Hence $f(n) \neq T$ in this case too: $n \in f(n)$, $n \notin T$.

Hence in all cases, $f(n) \neq T$. Since n was arbitrary, we have $f(n) \neq T$ for every $n \in \mathbb{N}$.

But now the claim follows: $T \notin \mathrm{Im} f$, hence f is not surjective.

Since f was arbitrary, there is no surjection $f: \mathbb{N} \to P(\mathbb{N})$ (hence no bijection)

Hence $\mathbb{N} \not\approx P(\mathbb{N})$.

The same proof works in general:

Theorem For any set A, there is no surjection $f: A \to P(A)$

Pf: Fix $f: A \to P(A)$ and let $T = \{a \in A \mid a \notin f(a)\}$

Then $\forall a \in A$, $f(a) \neq T$ (by same arg).

However: it's always the case that
$A \precsim P(A)$ ($f(a) = \{a\}$ defines an injection)
So above theorem shows that

$$A < P(A) \quad \text{for every } A.$$

It follows that there are infinitely many levels of infinity!

$$\mathbb{N} < P(\mathbb{N}) < P(P(\mathbb{N})) < \cdots$$

<u>Def'n</u> If X is infinite and $\mathbb{N} \not\approx X$ we say X is <u>uncountable</u>.

So, by above: $P(\mathbb{N})$ is unctbl.
Other examples?

<u>Sets of functions</u>: Consider the set F of functions $f : \mathbb{N} \to \{0,1\}$.

i.e. $F = \{ f \subseteq \mathbb{N} \times \{0,1\} \mid f \text{ is a function} \}$

↳ we can think of a given $f \in F$ as an infinite 01-sequence.

e.g. if $f(1) = 0$, $f(2) = 0$, $f(3) = 1$, $f(4) = 0$, $f(5) = 1$, $f(6) = 0$, $\vdots$

can picture f like this:
f "=" $001010\ldots$

Conversely could write:

g " = " 101010...

to mean that g is the function

$g(1) = 1$
$g(2) = 0$
$g(3) = 1$
$g(4) = 0$
$\vdots$ etc.

Theorem F is uncountable.

pf: Diagonalize!

Claim: if $H: \mathbb{N} \to F$ is a function then H is not a surjection.

pf: consider the function $f \in F$ defined as follows:

$$f(n) = \begin{cases} 1 & \text{iff } H(n)(n) = 0 \\ 0 & \text{iff } H(n)(n) = 1 \end{cases}$$

Then, by the very def'n of f we have that $(\forall n \in \mathbb{N})$ $f(n) \neq H(n)(n)$

hence: $(\forall n \in \mathbb{N})$ $f \neq H(n)$

f and $H(n)$ differ in the nth output

the claim follows ✓

To illustrate:

e.g. if we have:

$H(1)(1)$, $H(2)(2)$, $H(3)(3)$, $H(4)(4)$

$H(1) = 0\ 1\ 0\ 1\ 0 \ldots$

$H(2) = 0\ 0\ 1\ 1\ 0\ 1 \ldots$

$H(3) = 1\ 1\ 1\ 1\ 1\ 1 \ldots$

$H(4) = 0\ 0\ 0\ 1\ 1\ 1 \ldots$

$\vdots$

In this case would have

$f = 1100\ldots$

Observe: $f \neq H(n)$ for any n! **Why**: $f(n) \neq H(n)(n)$.