(4) Combining (2) and (3) gives $\mathbb{Z} \sim -\mathbb{N}$, (4) by transitivity of $\sim$.

<u>Def'n</u>: Let A, B be sets.

(1) We write $A \lesssim B$ (or $|A| \leq |B|$) to mean: there is an <u>injection</u> $f: A \to B$.

(2) We write $A \gtrsim B$ (or $|A| \geq |B|$) to mean: there is a <u>surjection</u> $g: A \to B$.

↪ we write $A < B$ to mean: there is an injection $f: A \to B$ but no bijection $g: A \to B$ (i.e. $A \lesssim B$ but $A \not\sim B$).

↪ similarly for $A > B$.

$\boxed{\text{NOTE!}}$ $A \gtrsim B$ is <u>not</u> simply the "reverse of" $A \lesssim B$, i.e. it is <u>not</u> asserting literally there is an injection from B to A.

<u>But</u>: this follows!

<u>Theorem</u>. For all sets A, B we have:

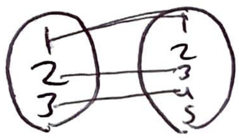
$A \lesssim B$ iff $B \gtrsim A$

i.e. there is an injection $f: A \to B$ iff there is a surjection $g: B \to A$.

Before proving theorem, let's illustrate the ideas w/ examples. (5)

<u>ex</u>: (1) Consider $f: [3] \to [5]$

defined by: $f(1) = 1$ $f(2) = 3$ $f(3) = 4$

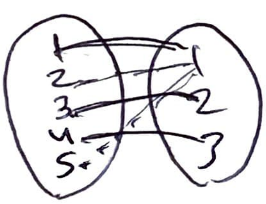


<u>Observe</u>: f is an injection, hence $[3] \lesssim [5]$.

<u>Idea</u>: to show $[5] \gtrsim [3]$, i.e. to get a surjection $g: [5] \to [3]$ we can "reverse" f then map anything left over to something arbitrary.

e.g. let: $g(1) = 1$, $g(3) = 2$, $g(4) = 3$ } "reverse" of f

$g(2) = 1$, $g(5) = 1$ } could put anything here.



then: g is a <u>function</u> since f was injective, and is clearly surjective.

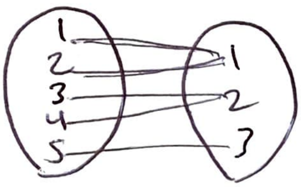
more generally can prove:

if $A \lesssim B$ then $B \gtrsim A$

using this idea.

② Consider: $g : [5] \to [3]$ defined by:

$g(1) = g(2) = 1$
$g(2) = g(3) = 2$
$g(5) = 3$

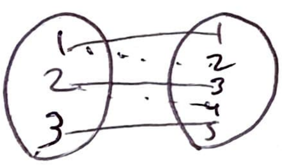


observe: g is a surjection hence $[5] \gtrsim [3]$.

<u>idea</u>: to show $[3] \lesssim 5$, i.e. get an <u>injection</u> $f : [3] \to [5]$ we take some "reverse" of g w/o repeats.

e.g. define $f : [3] \to [5]$ by:

$f(1) = 1 \quad f(2) = 3 \quad f(3) = 5$



"reverse" g — w/o repeats!

<u>then</u>: f is a function because g was surjective + we deleted repeats

and it's injective because g was a function.

more generally can prove: if $B \gtrsim A$ then $A \lesssim B$ using same idea.

Now let's prove theorem:

Pf ($\Rightarrow$) Sps $A \lesssim B$, i.e. $\exists f: A \to B$ an injection.

WTS: $B \gtrsim A$, i.e. $\exists g: B \to A$ a surjection

Define g as follows:

- first, fix some $a_0 \in A$
- now consider a given $b \in B$.

↳ if $b \in \mathrm{Im} f$, then $\exists a \in A$ s.t. $f(a) = b$. Moreover this a is *unique*, since f is injective. In this case define $g(b) = a$.

↳ if $b \notin \mathrm{Im} f$, define $g(b) = a_0$.

Then: ① g is a function from B to A

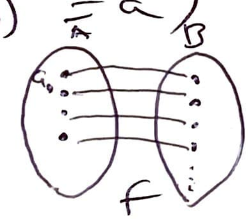
(**Why**: if $b \in \mathrm{Im} f$ then $g(b) =$ unique $a \in A$ s.t. $f(a) = b$

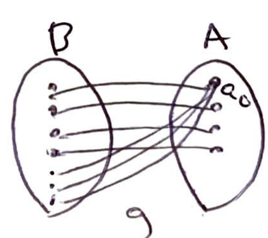
if $b \notin \mathrm{Im} f$ then $g(b) = a_0$)

② g is surjective

(**Why**: Fix $a \in A$. Let $b = f(a)$. Then by def'n $g(b) = a$)

Picture:





(8)

Now suppose $B \gtrsim A$, i.e. there is a surjection $g: B \to A$.

<u>WTS</u>: $A \lesssim B$, i.e. there is an injection $f: A \to B$.

Define f as follows:

- given $a \in A$, since g is surjective there is at least one $b \in B$ s.t. $g(b) = a$, i.e. $\text{PreIm}_g(\{a\}) \neq \emptyset$.
- So: <u>pick</u> <u>one</u> distinguished el't $b_a \in \text{PreIm}_g(\{a\})$ and define $f(a) = b_a$.

(Note: $g(b_a) = a$).

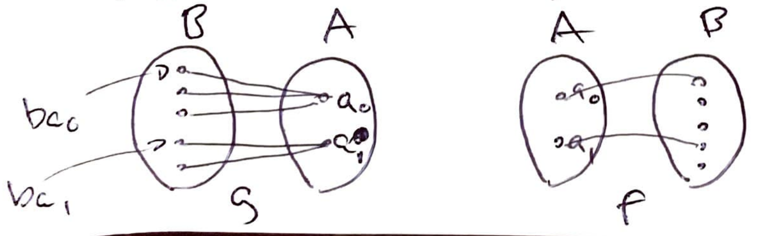
$\to$ do this for every $a \in A$.

<u>Then</u>: ① f is a function from A to B.

(<u>Why</u>: every $a \in A$ has a unique output $b_a \in B$ (from $\text{PreIm}_g(\{a\})$))

② f is injective: if $f(a) = f(a') = b$ then by def'n of f we have $g(b) = a$ and $g(b) = a'$.

Hence $a = a'$, since g is a function



Properties of $\lesssim$ and $\gtrsim$:

Sps A, B, C are sets.

① $A \lesssim A$ and $A \gtrsim A$ since $id_A : A \to A$ is both an injection and surjection. Hence $\lesssim$ and $\gtrsim$ are reflexive.

② If $A \lesssim B$ and $B \lesssim C$ then $\exists f: A \to B$ and $\exists g: B \to C$ injections. By HW: $g \circ f: A \to C$ is an injection. Hence $A \lesssim C$.

Sim'ly if $A \gtrsim B \gtrsim C$ then $A \gtrsim C$.

So $\lesssim$ and $\gtrsim$ are transitive.

③ Are they antisymmetric? Not literally. If $A \lesssim B$ and $B \lesssim A$, not necessarily $A = B$.

e.g. $A = \{1, 2, 3\}$ $B = \{*, \heartsuit, \Delta\}$.

$\boxed{\text{But!!}}$ turns out in this case that $A \sim B$ always (more on this later)

④ Are they total? That is, given sets A, B do we always have $A \lesssim B$ or $B \lesssim A$?

Yes! If you assume the axiom of choice (beyond our scope).

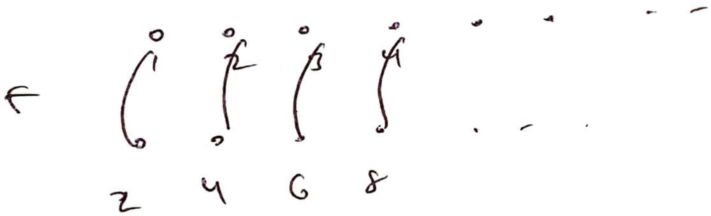
Some paradoxes of infinity:

Theme: $A \sim B$ when A, B are infinite can be counterintuitive!

① Let $E = \{2, 4, 6, 8, \ldots\}$

Thm: $\mathbb{N} \sim E$ ("there are as many even numbers as whole numbers")

Why: $f : \mathbb{N} \to E$ defined by $f(n) = 2n$ is a bijection



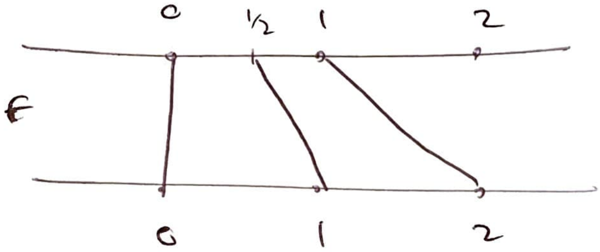
② Let $S = \{1, 4, 9, 16, \ldots\}$ = set of squares

Thm: $\mathbb{N} \sim S$ ("there are as many squares as whole numbers")

Why: $f : \mathbb{N} \to S$ defined by $f(n) = n^2$ is a bijection.

③ We have $[0,1] \sim [0,2]$. ("There are as many real numbers between 0 and 1 as between 0 and 2".)

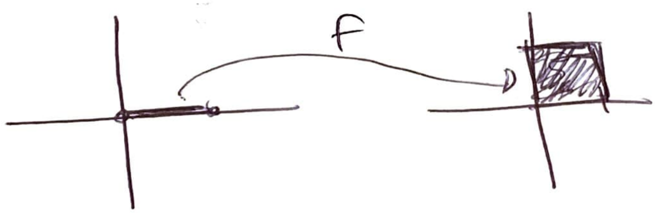
<u>Why</u>: $f : [0,1] \to [0,2]$
defined by $f(x) = 2x$ is a bijection.



(4) And actually: "The side is as large as the square"

i.e. $[0,1] \sim [0,1] \times [0,1]$
i.e. there is a bijection
$f : [0,1] \to [0,1] \times [0,1]$

<u>Why</u>: beyond our scope.



<u>moral</u>: infinite sets are wild.

Finiteness + Infiniteness

Def'n ① A set X is finite iff either

(i) $X = \emptyset$

or (ii) $\exists n \in \mathbb{N}$ and a bijection $f: [n] \to X$.

② X is infinite iff it is not finite, i.e.

(i) $X \neq \emptyset$

and (ii) $\forall n \in \mathbb{N}$ there is no bijection $f: [n] \to X$.

Following theorem says: $\mathbb{N}$ has the "smallest" possible infinite size.

Theorem If a set X is infinite, then $\mathbb{N} \lesssim X$.

Pf: Sps X is infinite. We define an injection $f: \mathbb{N} \to X$ inductively.

(BC) Since $X \neq \emptyset$ there is some $x_1 \in X$.

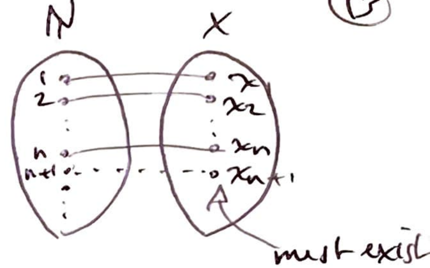
Let $f_1 = \{(1, x_1)\}$

(IH) Sps at stage n we have defined a "partial injection" f_n, i.e.

$f_n = \{(1, x_1), (2, x_2), \ldots, (n, x_n)\}$

and the x_i's are all distinct

(IS) We can view f_n as an injection from $[n]$ to X. Since X is not finite, must be that f_n is not a surjection (o/w it would be a bijection)



i.e. $\exists x_{n+1} \in X$ s.t. $x_{n+1} \notin \{x_1, x_2, \ldots, x_n\}$

So define $f_{n+1} = \{(1,x_1),(2,x_2),\ldots,(n,x_n),(n+1, x_{n+1})\}$

$= f_n \cup \{(n+1, x_{n+1})\}$.

Continuing inductively we can define a sequence of injections $f_n : [n] \to X$ s.t. if $n \le m$ then $f_n \subseteq f_m$.

Now: Let $f = \bigcup_{n \in \mathbb{N}} f_n = \{(1,x_1),(2,x_2),\ldots\}$

then $f : \mathbb{N} \to X$ is an injection (why: we arranged all the x_i's distinct).

Note: proof is a bit informal; we're using not only induction, but implicitly what set theorists call the recursion theorem, and the axiom of choice.