ex's Define $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by

$$f(m,n) = m+n$$

$g: \mathbb{Z} \to \mathbb{N}$ by

$$g(n) = n^2 + 1$$

then: $g \circ f(1,3) = g(f(1,3))$
$= g(1+3)$
$= g(4) = 4^2 + 1 = 17$

In general: $g \circ f(m,n) = g(f(m,n))$
$= g(m+n)$
$= (m+n)^2 + 1$

observe $g \circ f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$.

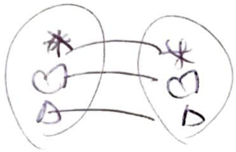
The identity function:

Def'n Sps A is a set. The identity function on A, denoted id_A is defined by:

$$id_A : A \to A$$
$$id_A(x) = x$$

e.g. if $A = \{*, ♡, \Delta\}$ then $id_A : A \to A$ is

$$id_A = \{(*,*), (♡,♡), (\Delta,\Delta)\}$$



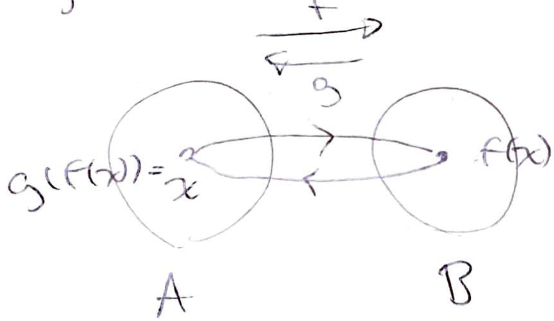
<u>Def'n</u>: a function $f: A \to B$ is called <u>invertible</u> iff there is a function $g: B \to A$ s.t.

$$g \circ f = id_A \quad \text{i.e. } (\forall x \in A)\ g(f(x)) = x$$

and

$$f \circ g = id_B \quad \text{i.e. } (\forall y \in B)\ f(g(y)) = y$$

when g exists, it is called the <u>inverse</u> <u>of f</u>, and denoted f^{-1}



<u>Ex</u>: Consider $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$.

Let $g: \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \frac{x-1}{2}$$

Observe: $(\forall x \in \mathbb{R})$ $g(f(x)) = g(2x+1)$

$$= \frac{(2x+1)-1}{2}$$

$$= x$$

hence: $g \circ f = id_{\mathbb{R}}$

also: $(\forall x \in \mathbb{R})$ $f(g(x)) = f\left(\frac{x-1}{2}\right)$

$$= 2\left(\frac{x-1}{2}\right) + 1$$

$$= x$$

hence $f \circ g = id_{\mathbb{R}}$.

thus: f is invertible - it's inverse f^{-1} is g.

Note: not all functions are invertible!

In fact:

Theorem a function $f: A \to B$ is invertible iff f is a bijection

Pf: $(\Rightarrow)$ Suppose f is invertible and let $g = f^{-1}$ be its inverse.

WTS: f is a bijection

(surjectivity): - fix $y \in B$

- let $x = g(y)$

- then: $f(x) = f(g(y))$

$= y$ $(f \circ g = id_B)$

Since $y \in B$ was arbitrary, f is surjective ✓

(injectivity): - Fix $x, y \in A$ and suppose $f(x) = f(y)$

- then $g(f(x)) = g(f(y))$

- hence $x = y$ $(g \circ f = id_A)$

Since $x, y \in A$ were arbitrary, f is injective ✓

($\Leftarrow$) Suppose now that $f: A \to B$ is a bijection

WTS: f is invertible

define: $g = \{(b,a) \in B \times A \mid (a,b) \in f\}$

Claim ① g is a function from B to A

② $g = f^{-1}$

Pf ① WTS: $\forall b \in B$ $\exists$ a unique $a \in A$ s.t. $(b,a) \in g$.

- So fix $b \in B$.

(existence): - Since f is surjective, $\exists a \in A$ s.t. $f(a) = b$, i.e. $(a,b) \in f$.

- hence $(b,a) \in g$, by def'n of g

(uniqueness): - Sps there is $a' \in A$ s.t. $(b,a') \in g$ as well.

- then must be that $(a',b) \in f$
- i.e. $f(a') = b$
- but then $f(a') = f(a)$. Since f is injective, this implies $a = a'$.

Hence ① is proved ✓

② WTS: $g = f^{-1}$ i.e. $g \circ f = id_A$, $f \circ g = id_B$.

- So fix $a \in A$.
- Let $b = f(a)$, so that $(a,b) \in f$.
- then $(b,a) \in g$, i.e. $g(b) = a$

hence $g(f(a)) = g(b) = a$.

- Since a was arbitrary, $g \circ f = id_A$.

- Now fix $b \in B$. Let $a = g(b)$, so that $(b,a) \in g$.
- then must be that $(a,b) \in f$, i.e. $f(a) = b$.
- hence $f \circ g(b) = f(g(b)) = f(a) = b$.
- since b was arbitrary, $f \circ g = id_B$ ✓

hence $g = f^{-1}$, as claimed.

We can sometimes use thm to prove a given f is a bijection

ex: $f : \mathbb{R} \to \mathbb{R}$

$f(x) = 2x+1$ is a bijection

Pf: we checked above: if $g(x) = \frac{x-1}{2}$

then $g = f^{-1}$

hence f is invertible - hence (by thm) a bijection (of $\mathbb{R}$ w/ itself)

Infinity:

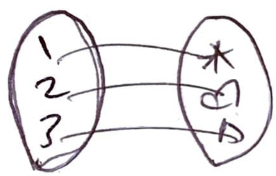
The concept of cardinality:

- we would say that {*, ♡, Δ} has 3 el'ts, or is of size 3.

- why? By counting it!

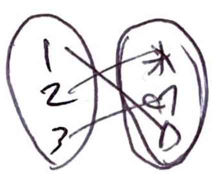
  {*, ♡, Δ}
  1 2 3

- in doing so we are implicitly defining a bijection between {1,2,3} and {*, ♡, Δ}



- we could've counted differently:

  {*, ♡, Δ}
  2 3 1



- generalizing this idea: we'll say two sets have the same size iff there is a bijection between them.

Def'n We say that two sets A, B have the <u>same cardinality</u>, and write $A \sim B$, iff there is a bijection $f: A \to B$.

Note: - In set theory courses, one defines, for every set A, the cardinal number $|A|$.

- Can then prove: $A \sim B$ iff $|A| = |B|$
(e.g. $|\{*, \heartsuit, \circ\}| = |\{\circ, \square, \triangle\}| = 3$)

- defining cardinal #s beyond our scope: For us: $|A| = |B|$ just means

$A \sim B$

i.e. $\exists f: A \to B$ a bijection.

Properties of $\sim$: ① For any set A, $id_A: A \to A$ is a bijection (why?)

- Hence $A \sim A$, i.e. $\sim$ is <u>reflexive</u>.

② If $f: A \to B$ is a bijection, then f is invertible and $f^{-1}: B \to A$ is a bijection too (why?)

- Hence if $A \sim B$ then $B \sim A$.

- i.e. $\sim$ is symmetric.

3 On HW you show: if $f: A \to B$, $g: B \to C$ are bijections then $g \circ f: A \to C$ is a bijection.

- Hence if $A \sim B$ and $B \sim C$ then $A \sim C$, i.e. $\sim$ is <u>transitive</u>.

↳ 1 + 2 + 3 show $\sim$ is an equiv. relation on sets!

↳ $\sim$ most interesting when the sets being compared are <u>infinite</u>.

<u>Ex's</u>: 1 $\{1,2,3\} \sim \{*, \heartsuit, \Delta\}$ since $f = \{(1,*), (2,\heartsuit), (3,\Delta)\}$ is a bijection.

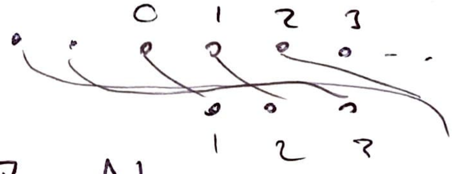
2 Let $-\mathbb{N}$ denote the set $\{-1,-2,-3,\ldots\}$ define $f: \mathbb{N} \to -\mathbb{N}$ by:

$$f(n) = -n$$

- ez to check: f is a bijection.
- Hence $\mathbb{N} \sim -\mathbb{N}$.

3 we showed before: $f: \mathbb{Z} \to \mathbb{N}$ defined by

$$f(n) = \begin{cases} 2n & n > 0 \\ 2(-n)+1 & n \leq 0 \end{cases}$$



is a bijection. Hence $\mathbb{Z} \sim \mathbb{N}$.