Again we have strict containment in this case:

$$Im_f(PreIm_f(Y)) = \{1\} \subsetneq \{-2,1\} = Y.$$

-jections

Let $A = \{1,2,3\}$

$B = \{*, \heartsuit\}$

$C = \{1,2\}$

$D = \{*, \heartsuit, \triangle\}$

define $g : A \to B$

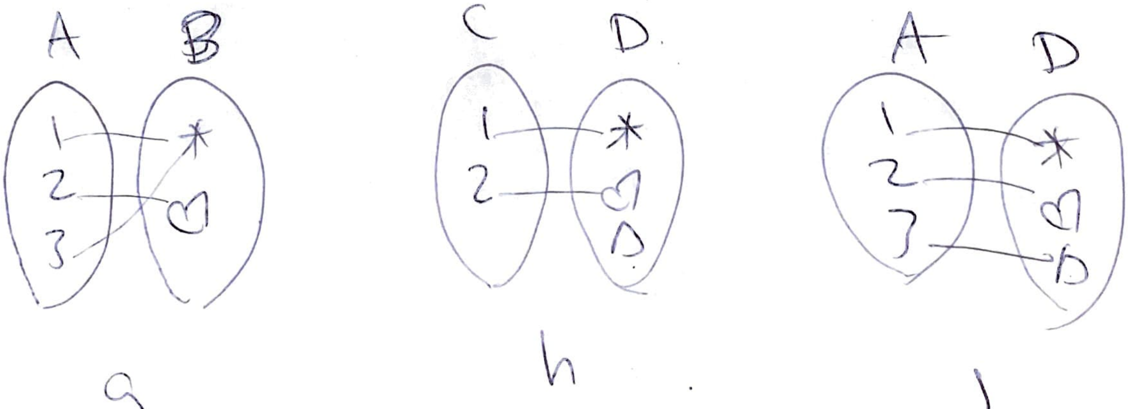
$h : C \to D$

$j : A \to D$

by: $g = \{(1,*), (2,\heartsuit), (3,*)\}$

$h = \{(1,*), (2,\heartsuit)\}$

$j = \{(1,*), (2,\heartsuit), (3,\triangle)\}$.



g $\qquad$ h $\qquad$ j

<u>Def'n</u> a function $f: A \to B$ is <u>surjective</u> (or <u>onto</u>, or a <u>surjection</u>) iff $Im_f = B$

i.e. iff

$$(\forall b \in B)(\exists a \in A)(f(a) = b)$$

<u>ex's</u> - g and j above are surjective
- h is not, since $\Delta \notin Im_h$

<u>Proving</u> <u>surjectivity</u>:

<u>ex</u>: ① Define $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $f((m,n)) = m+n$.
write $f(m,n)$

<u>Claim</u> f is surjective

<u>Pf</u>: <u>WTS</u> $(\forall x \in \mathbb{Z})(\exists (a,b) \in \mathbb{Z} \times \mathbb{Z})(f(a,b) = x)$

- so fix $x \in \mathbb{Z}$
- observe $f(0,x) = 0 + x = x$
- hence $\exists (a,b) \in \mathbb{Z} \times \mathbb{Z}$ s.t. $f(a,b) = x$
namely $(a,b) = (0,x)$.
- since x was arbitrary, claim is proved ✓

(15)

② Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x+1$.

Claim: f is surjective

Pf: - Fix $y \in \mathbb{R}$.

- Let $x = \frac{y-1}{2}$

- Then $f(x) = f\left(\frac{y-1}{2}\right) = 2\left(\frac{y-1}{2}\right)+1 = y$

- Since y was arbitrary, claim is proved.

③ Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

Claim: f is not surjective

Pf: WTS $\neg(\forall y \in \mathbb{R})(\exists x \in \mathbb{R})(f(x) = y)$

i.e. $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(f(x) \neq y)$

Let $y = -1$: Then fix $x \in \mathbb{R}$.

Observe $f(x) = x^2 \geq 0 > -1$

$\Rightarrow f(x) \neq -1$.

Hence, since x was arbitrary,

$(\forall x \in \mathbb{R})(f(x) \neq -1)$

Hence $-1 \notin \operatorname{Im} f \Rightarrow f$ is not surjective

Def'n a function $f: A \to B$ is <u>injective</u> (or <u>one-to-one</u>, or <u>1-1</u>, or <u>an injection</u>) iff

$$(\forall x, y \in A)\,(f(x) = f(y) \Rightarrow x = y)$$

equivalently:

$$(\forall x, y \in A)\,(x \neq y \Rightarrow f(x) \neq f(y))$$

"distinct inputs map to distinct outputs"

<u>ex's</u> - g above is <u>not</u> injective since $1 \neq 3$ but $g(1) = g(3) = *$

- h, j are injective.

<u>Proving</u> <u>injectivity</u>:

<u>Two approaches</u>: Fix $x, y \in A$ and either:

1. Assume $f(x) = f(y)$, prove $x = y$
2. Assume $x \neq y$, prove $f(x) \neq f(y)$

<u>Ex's</u> ① Consider again $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x+1$.

Claim: f is injective.

<u>Pf</u>: - fix $x, y \in \mathbb{R}$

- assume $f(x) = f(y)$

i.e. $2x+1 = 2y+1$

$\Rightarrow 2x = 2y$

$\Rightarrow x = y$ ✓ since x,y arbitrary, claim is proved

(2) Define $f: \mathbb{N} \to \mathbb{N}$ by $f(n) = n^2$

Claim: f is injective.

Pf: fix $n, m \in \mathbb{N}$ and sps $n \neq m$

WTS $f(n) \neq f(m)$

Two cases: (1) $m < n$

(2) $n < m$

If (1): since n, m both positive can square both sides of inequality to get:

$m^2 < n^2$

so in particular $f(m) \neq f(n)$

If (2): similar.

Since n, m were arbitrary, claim is proved.

③ Define: $f: \mathbb{Z} \to \mathbb{Z}$ by $f(n) = n^2$

Claim: f is not injective

PF: $f(-2) = f(2) = 4$

but $2 \neq -2$.

Def'n: A function $f: A \to B$ is bijective (or a bijection) iff f is both injective and surjective.

ex's: - g above is not bijective (surjective, but not injective)

- nor is h (injective, but not surjective)

- j is bijective.

Proving bijectivity:

Ex's: ① Consider again $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x+1$.

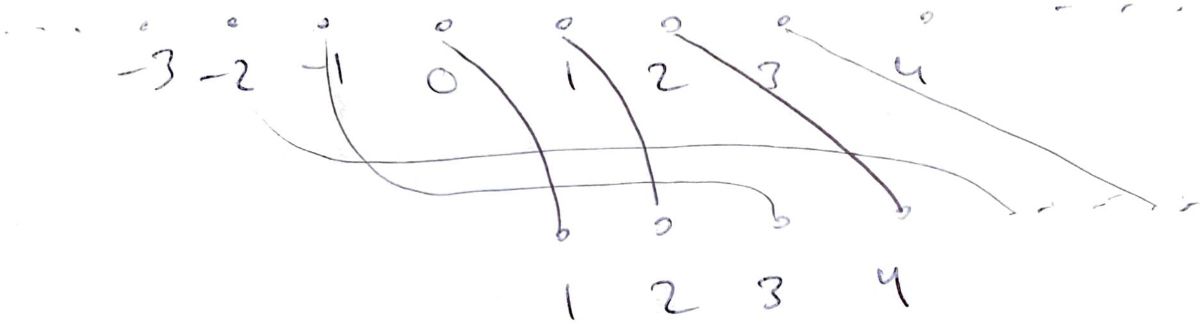
Claim f is bijective

Pf: we've already shown f is both injective and surjective.

② A spicy one: define $f: \mathbb{Z} \to \mathbb{N}$ by: 52

$$f(n) = \begin{cases} 2n & \text{if } n > 0 \\ 2(-n)+1 & \text{if } n \leq 0 \end{cases}$$

Picture:



Claim f is bijective.

Pf: (surjectivity):

- Fix $n \in \mathbb{N}$
- if n is even, then $n = 2k$ for some $k \in \mathbb{N}$ (hence $k > 0$)
- hence $f(k) = 2k = n$
- if n is odd, then $n = 2k+1$ for some $k \in \mathbb{N} \cup \{0\}$ (hence $k \geq 0$, hence $-k \leq 0$)
- hence $f(-k) = 2k+1 = n$

- In either case: $(\exists x \in \mathbb{Z})(f(x) = n)$
- hence f is surjective ✓

(injectivity)

- Fix $n, m \in \mathbb{Z}$ and assume $n \neq m$.

We WTS $f(n) \neq f(m)$

- We will assume $n < m$, since case when $m < n$ is similar.

<u>Case 1:</u> $0 < n < m$.

- then $f(n) = 2n < 2m = f(m)$
- hence $f(n) \neq f(m)$

<u>Case 2:</u> $n < m \leq 0$.

- then $f(n) = 2(-n) + 1$
  $f(m) = 2(-m) + 1$
- <u>observe:</u> Since $n < m$

$$-n > -m$$
$$\Rightarrow 2(-n) + 1 > 2(-m) + 1$$

i.e. $f(n) > f(m)$

hence $f(n) \neq f(m)$.

<u>Case 3</u> $n \leq 0 < m$

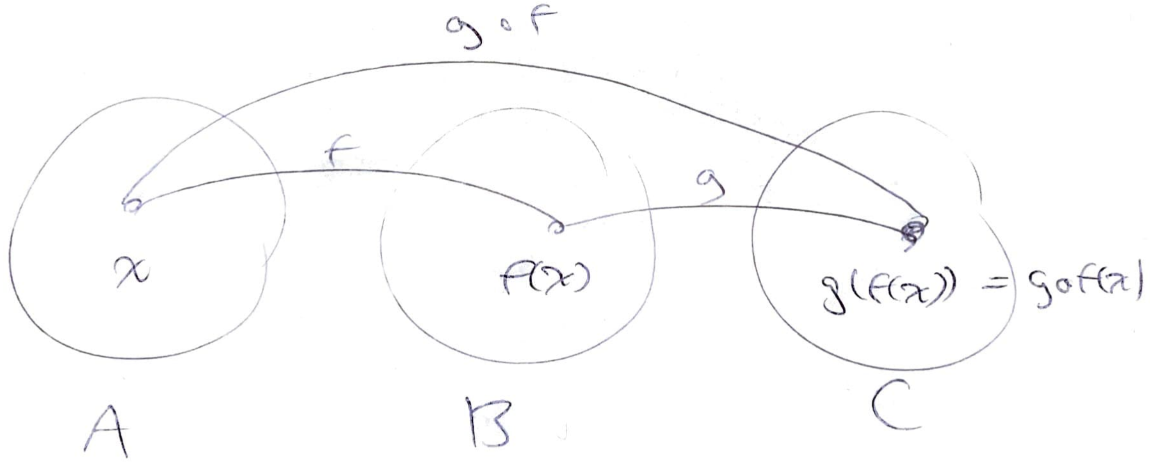
- then $f(n) = 2(-n) + 1$ is odd
  $f(m) = 2m$ is even.

- hence $f(n) \neq f(m)$ in this case as well.
- hence in all cases $f(n) \neq f(m)$
- Since n, m were arbitrary, we've proved f is injective
- hence f is bijective ✓

Compositions:

Def'n Spse $f: A \to B$ and $g: B \to C$ are functions. The composition of f and g, denoted $g \circ f$, is defined by:

$$(\forall x \in A) \quad g \circ f(x) = g(f(x))$$



Observe: So defined, we see that $g \circ f$ is a function from A to C.