then: $f(1) = 12 = g(1)$
$f(2) = 30 = g(2)$
$f(3) = 60 = g(3)$ hence $f = g$!

i.e. $f = \{(1,12), (2,30), (3,60)\} = g$.

[What's the magic trick?

observe: $f - g = x^3 - 6x^2 + 11x - 6$
$= (x-1)(x-2)(x-3)$
$= 0$ for $x \in \{1,2,3\}$]

Images

Def'n Spse $f: A \to B$ is a function and $X \subseteq A$. The image of X under f, denoted $\mathrm{Im}_f(X)$, is defined as:

$$\mathrm{Im}_f(X) = \{b \in B \mid (\exists a \in X)\ f(a) = b\}$$

informally we could write,

$$\mathrm{Im}_f(X) = \{f(a) \mid a \in X\}$$

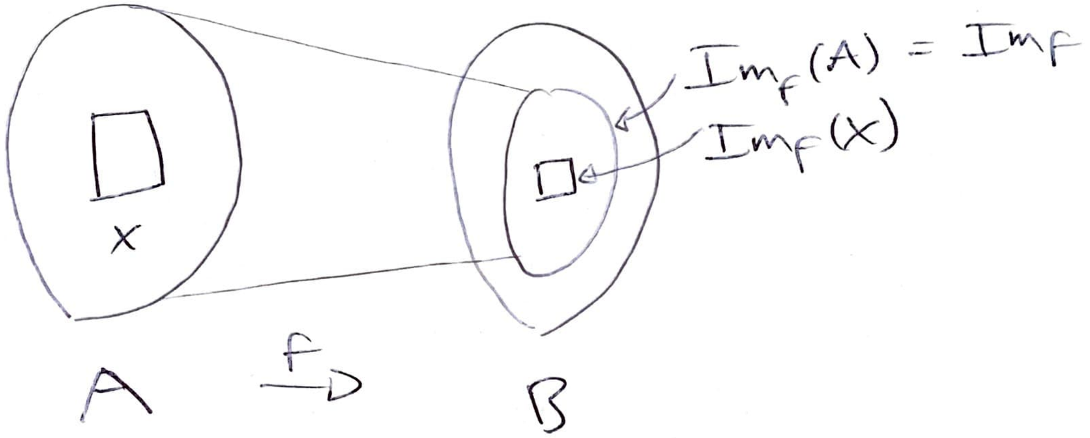
the point is: $a \in X \Leftrightarrow f(a) \in \mathrm{Im}_f(X)$

Note: when $X = A$ itself we say $\mathrm{Im}_f(A)$ is the image of f and sometimes just write Im_f.

So def'n says: - $Im_f(X)$ is the "set of outputs of elements in X"

- $Im_f = Im_f(A)$ is the "set of all outputs of f."

Pic:



Ex's ① Let $A = \{1,2,3\}$ $B = \{*, \heartsuit, \Delta\}$

define $f: A \to B$ by $f = \{(1,*), (2,\heartsuit), (3,*)\}$

Then: - $Im_f(\{1,3\}) = \{f(1), f(3)\}$
$= \{*, *\}$
$= \{*\}$

and: - $Im_f = Im_f(\{1,2,3\}) = \{f(1), f(2), f(3)\}$
$= \{*, \heartsuit, *\}$
$= \{*, \heartsuit\}$.

② Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$

Then: - $\mathrm{Im}_f(\{-1,0,1\}) = \{(-1)^2, 0^2, 1^2\}$
$= \{0, 1\}$

- $\mathrm{Im}_f = \{x \in \mathbb{R} \mid x \geq 0\}$

Functions add a layer of complexity to the basic set theory of $\cup, \cap, \ldots$ we studied before.

Prop'n: Sps $f: A \to B$ is a function and $S, T \subseteq A$.
Then $\mathrm{Im}_f(S \cap T) \subseteq \mathrm{Im}_f(S) \cap \mathrm{Im}_f(T)$

PF: - fix $y \in \mathrm{Im}_f(S \cap T)$
- then $\exists x \in S \cap T$ s.t. $f(x) = y$.
- hence $x \in S$ and $x \in T$
- hence $f(x) \in \mathrm{Im}_f(S)$ and $f(x) \in \mathrm{Im}_f(T)$
- i.e. $y \in \mathrm{Im}_f(S)$ and $y \in \mathrm{Im}_f(T)$
- i.e. $y \in \mathrm{Im}_f(S) \cap \mathrm{Im}_f(T)$

Since y was arbitrary the prop'n is proved ✓

Note: in general we don't have:

$$\mathrm{Im}_f(S \cap T) = \mathrm{Im}_f(S) \cap \mathrm{Im}_f(T)$$

e.g. consider $f(x) = x^2$ on $\mathbb{R}$.

Let $S = \{-1, 0\}$ $\quad T = \{0, 1, 2\}$

Then: $\mathrm{Im}_f(S) = \{f(-1), f(0)\} = \{1, 0\}$

$\mathrm{Im}_f(T) = \{f(0), f(1), f(2)\} = \{0, 1, 4\}$

$\Rightarrow \mathrm{Im}_f(S) \cap \mathrm{Im}_f(T) = \{0, 1\}$

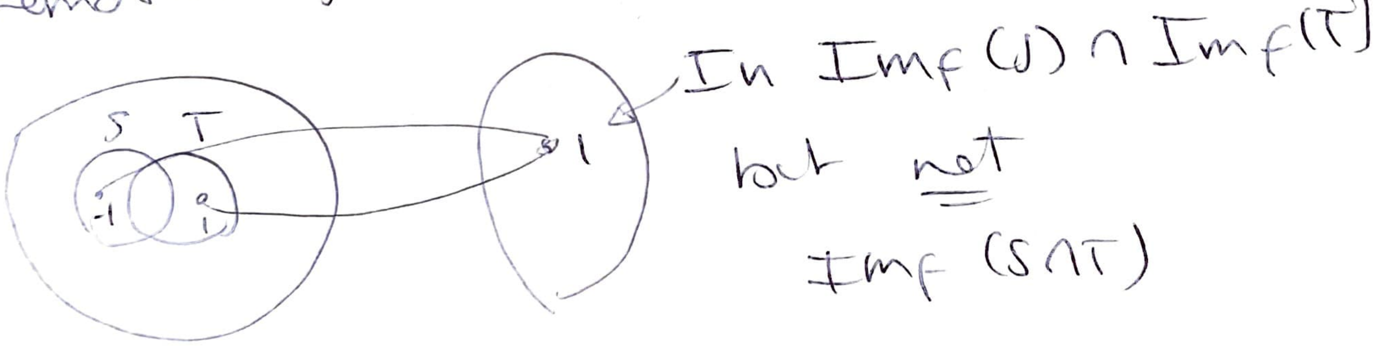
OTOH: $\mathrm{Im}_f(S \cap T) = \mathrm{Im}_f(\{0\})$

$= \{f(0)\} = \{0\}$

So in this case: $\mathrm{Im}_f(S \cap T) \subsetneq \mathrm{Im}_f(S) \cap \mathrm{Im}_f(T)$

The essence of the issue: functions can send multiple inputs to the same output.

schematically:



Pre Images:

Def'n: Sps $f: A \to B$ is a function a $Y \subseteq B$. The preimage of y under f, denoted $\mathrm{PreIm}_f(Y)$, is defined as:

$$\mathrm{PreIm}_f(Y) = \{x \in A \mid f(x) \in Y\}$$

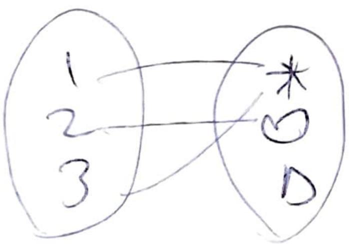
= the set of inputs in A whose outputs land in Y.

Note: since $f(x) \in B$ for every $x \in A$, we don't separately define $\mathrm{PreIm}_f(B)$ - this is always just A.

Ex: ① Let $A = \{1,2,3\}$

$B = \{*, \heartsuit, \Delta\}$

$f = \{(1,*), (2,\heartsuit), (3,*)\}$



Then: $\mathrm{PreIm}_f(\{*\}) =$

$\{x \in A \mid f(x) \in \{*\}\}$

$= \{x \in A \mid f(x) = *\}$

$= \{1,3\}$

(42)

$$\mathrm{PreIm}_f(\{*, ♡\}) = \{x \in A \mid f(x) \in \{*, ♡\}\}$$
$$= \{1,2,3\} = A$$

$$\mathrm{PreIm}_f(\{\Delta\}) = \{x \in A \mid f(x) \in \{\Delta\}\}$$
$$= \{x \in A \mid f(x) = \Delta\}$$
$$= \emptyset.$$

(2) Consider $f : \mathbb{R} \to \mathbb{R}$
$$f(x) = x^2$$

<u>Then;</u> $$\mathrm{PreIm}_f(\{0,1\}) = \{x \in \mathbb{R} \mid f(x) \in \{0,1\}\}$$
$$= \{x \in \mathbb{R} \mid x^2 \in \{0,1\}\}$$
$$= \{-1, 0, 1\}$$

$$\mathrm{PreIm}_f([0,2]) = \{x \in \mathbb{R} \mid x^2 \in [0,2]\}$$
$$= \{x \in \mathbb{R} \mid 0 \le x^2 \le 2\}$$
$$= \{x \in \mathbb{R} \mid x^2 \le 2\}$$
$$= \{x \in \mathbb{R} \mid -\sqrt{2} \le x \le \sqrt{2}\}$$
$$= [-\sqrt{2}, \sqrt{2}].$$

$$\mathrm{PreIm}_f([0,\infty)) = \{x \in \mathbb{R} \mid x^2 \in [0,\infty)\}$$
$$= \mathbb{R}.$$

Q: what happens if we take the preimage of the image of some $X \subseteq A$?
or the image of the preimage of some $Y \subseteq B$?

<u>Prop'n</u> Sps $f: A \to B$ is a function.

(i) Fix $X \subseteq A$.
<u>Then</u>: $\mathrm{PreIm}_f(\mathrm{Im}_f(X)) \supseteq X$

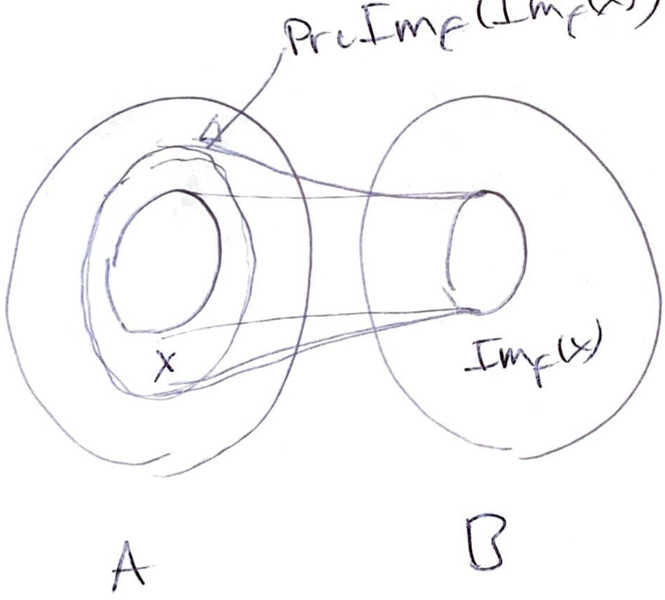
(ii) Fix $Y \subseteq B$
<u>Then</u>: $\mathrm{Im}_f(\mathrm{PreIm}_f(Y)) \subseteq Y$.

<u>Pf</u>: (i) Fix $x \in X$.
By def'n: $\mathrm{PreIm}_f(\mathrm{Im}_f(X))$
$$= \{y \in A \mid f(y) \in \mathrm{Im}_f(X)\}$$
but since $x \in X$, we know $f(x) \in \mathrm{Im}_f(X)$
by def'n of $\mathrm{Im}_f(X)$
Hence $x \in \mathrm{PreIm}_f(\mathrm{Im}_f(X))$
Since x was arbitrary, (i) is proved.

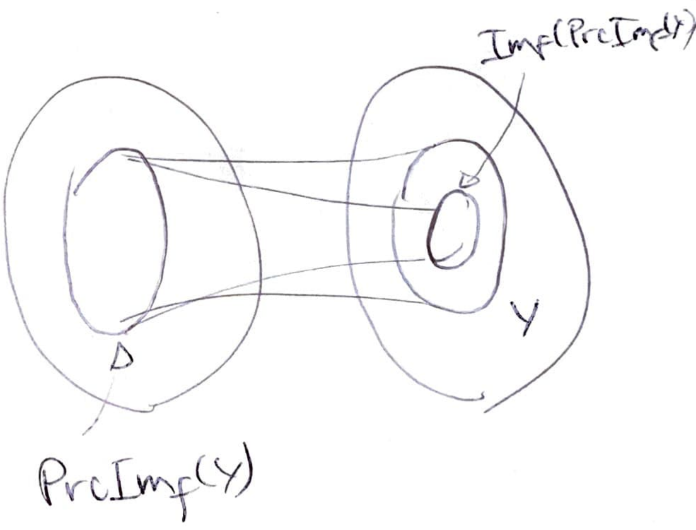
(ii) Now fix $y \in Im_f(PreIm_f(Y))$
by def'n of image, $\exists x \in PreIm_f(Y)$
s.t. $f(x) = y$.
But then, by def'n of preimage,
$f(x) \in Y$, i.e. $y \in Y$.
Since y was arbitrary, (ii) is proved ✓

Picture:

(i)



(ii)



Note: in general, neither containment can be reversed.

Ex: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by
$f(x) = x^2$

Let $X = \{1\}$

Then: $Im_f(X) = Im_f(\{1\})$
$= \{f(1)\}$
$= \{1^2\} = \{1\}$

So then: $PreIm_f(Im_f(X)) = PreIm_f(\{1\})$
$= \{x \in \mathbb{R} \mid f(x) \in \{1\}\}$
$= \{x \in \mathbb{R} \mid x^2 \in \{1\}\}$
$= \{x \in \mathbb{R} \mid x^2 = 1\}$
$= \{-1, 1\}$

So we have strict containment in this case:
$X = \{1\} \subsetneq \{-1,1\} = PreIm_f(Im_f(X))$

Now let $Y = \{-2, 1\}$

Then: $PreIm_f(Y) = \{x \in \mathbb{R} \mid f(x) \in \{-2,1\}\}$
$= \{x \in \mathbb{R} \mid x^2 \in \{-2,1\}\}$
$= \{-1, 1\}$

So then: $Im_f(PreIm_f(Y)) = Im_f(\{-1,1\})$
$= \{f(-1), f(1)\}$
$= \{(-1)^2, 1^2\}$
$= \{1\}.$

Again we have strict containment in this case:

$$Im_f(PreIm_f(Y)) = \{1\} \subsetneq \{-2,1\} = Y.$$

-jections : Let $A = \{1,2,3\}$

$B = \{*, ♡\}$

$C = \{1,2\}$

$D = \{*, ♡, \Delta\}$

define $g : A \to B$

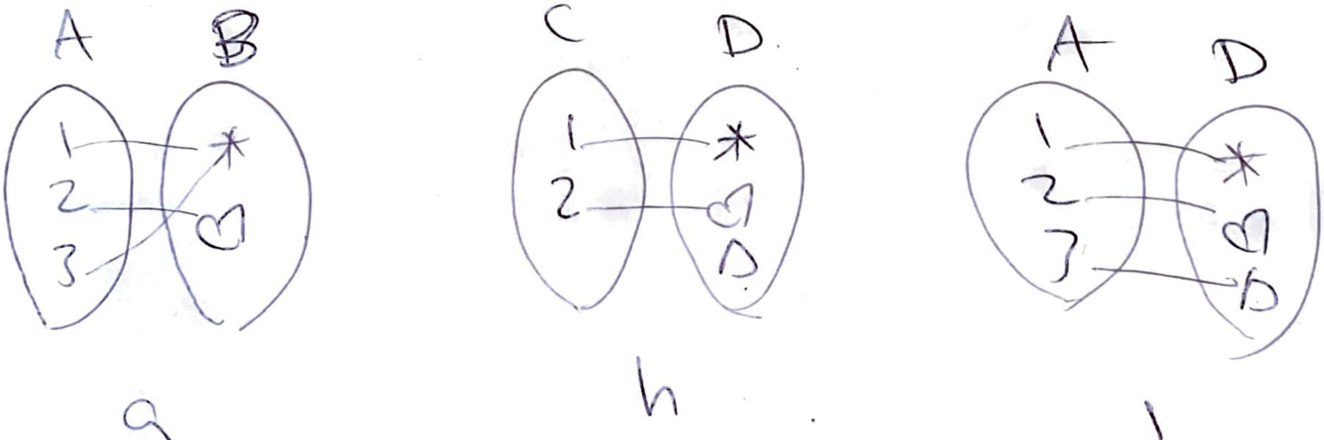
$h : C \to D$

$j : A \to D$

by: $g = \{(1,*), (2,♡), (3,*)\}$

$h = \{(1,*), (2,♡)\}$

$j = \{(1,*), (2,♡), (3,\Delta)\}$.



g h j