($\Leftarrow$) Now assume R is irreflexive and antisymmetric. Fix $x, y \in A$

We WTS: $(x,y) \in R \Rightarrow (y,x) \notin R$.

So assume $(x,y) \in R$. If it were the case that $(y,x) \in R$, then by antisymmetry this would imply $x = y$. Hence we would have $(x,x) \in R$ contradicting irreflexivity

Hence $(y,x) \notin R$, so asymmetry holds ✓

Prop'n shows that although official def'n of strict p.o. is a relation R that is (i) irrefl. (ii) transitive (iii) antisymm.

it's equiv to define such a relation as one which is (i) transitive (ii) asymmetric, and verifying these two properties often easier in practice.

e.g. to prove $<$ is a strict p.o. on $\mathbb{R}$ can instead just observe:

$\forall x, y, z \in \mathbb{R}$

(i) $x < y \wedge y < z \Rightarrow x < z$

(ii) $x < y \Rightarrow y \not< x$.

② Let A be a fixed set. Then $\subsetneq$ is a strict p.o. on $P(A)$.

PF: $\forall X, Y, Z \in P(A)$

(i) $X \subsetneq Y \wedge Y \subsetneq Z \Rightarrow X \subsetneq Z$

(ii) $X \subsetneq Y \Rightarrow Y \not\subsetneq X$. ✓

remexamples

① $\leq$ is not a strict partial order on $\mathbb{R}$ since $\leq$ is not irreflexive (in fact, $\leq$ is reflexive, which is stronger than not being irreflexive)

② OTOH, $<$ is not a (nonstrict) partial order on $\mathbb{R}$, since $<$ is not reflexive (in fact, $<$ is irreflexive)

③ More generally: a relation R cannot be both a strict and non-strict p.o. (unless $R = \emptyset$ is the empty relation).

④ $\neq$ (e.g. on $\mathbb{N}$) is neither a strict nor non-strict p.o. since $\neq$ is not transitive.

Total orders:

Def'n: A relation R on a set A is called total iff

$$(\forall x,y \in A)\big((x,y) \in R \lor (y,x) \in R \lor x = y\big)$$

Def'n (i) if R is a partial order on A that is also total, then R is called a total order on A.

(ii) if R is a strict partial order on A that is also total, then R is called a strict total order on A.

Ex's ① $\leq$ is a total order on $\mathbb{R}$: $\leq$ is a p.o. and further we know

$$(\forall x,y \in \mathbb{R})\,(x \leq y \lor y \leq x \lor x = y)$$

② $\subseteq$ is not a total order on $P(\mathbb{N})$: it is not total, e.g.

if $X = \{1,2\}$ $Y = \{2,3\}$

then $X \not\subseteq Y$ and $Y \not\subseteq X$ and $X \neq Y$

③ Similarly, $|$ is a p.o. on $\mathbb{N}$, but not a total order since e.g. $3 \nmid 5$ and $5 \nmid 3$ and $5 \neq 3$.

④ $<$ is a strict total order on $\mathbb{R}$:

We know it's a strict p.o. and it's total since

$(\forall x, y \in \mathbb{R})\ (x<y \lor y<x \lor x=y)$

⑤ $\subsetneq$ is not a strict total order on $P(\mathbb{N})$: it's not total.

Functions:

- Functions, like relations, are ubiquitous in math. But what "are" functions?
- intuitively: a function is a rule that assigns to each input x in a domain A a unique output $f(x)$ in a codomain B.
- idea: can define functions rigorously as a special type of relation.

Def'n a function (with domain A and codomain B) is a relation $f \subseteq A \times B$ s.t.:

for every $a \in A$
there is a unique $b \in B$
s.t. $(a,b) \in f$.

i.e. $(\forall a \in A)(\exists b \in B)\left[(a,b) \in f \land (\forall c \in B)\left((a,c) \in f \Rightarrow c=b\right)\right]$

<u>Notation</u>: • we write

$$f: A \to B$$

to mean that a relation $f \subseteq A \times B$ is a function.

• we also write

$$f(a) = b$$

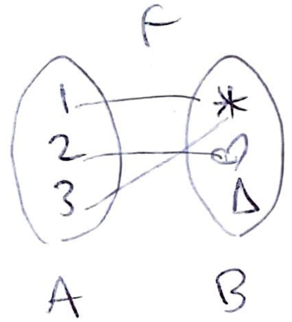
to mean $(a,b) \in f$.

<u>Note</u>: - def'n says: <u>every</u> $a \in A$ assigned an output $b \in B$.

- does <u>not</u> insist every $b \in B$ is mapped to by some input $a \in A$ (functions w/ this property are called <u>surjective</u>).

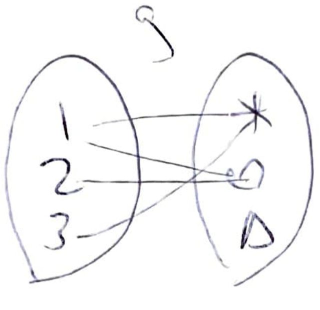
<u>Ex's</u> Let $A = \{1,2,3\}$ $B = \{*, ♡, \Delta\}$

Then: $f = \{(1,*), (2,♡), (3,*)\}$ is a function from A to B:



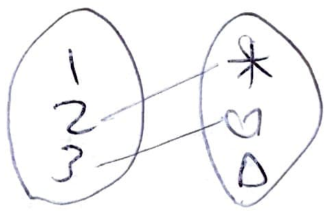
a function

but $g = \{(1,*), (1,♡), (2,♡), (3,*)\}$ is <u>not</u> a function since 1 does not have a unique output: both $(1,*)$ and $(1,♡) \in g$



not a function

nor is $h = \{(2,*), (3,\heartsuit)\}$ since 1 is not assigned an output



not a function

(both g, h are perfectly good relations)

② we'll often define functions by a rule:

e.g. $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

or $g: \mathbb{R} \to \mathbb{Z}$

$g(x) = \lfloor x \rfloor$

but behind the scenes these functions are still sets of ordered pairs

e.g. for f above we have:

$(2,4) \in f$ but $(7,12) \notin f$

$(3,9) \in f$ (since $f(7) = 49 \neq 12$)

Warning: not all rules yield well-defined functions!

e.g. suppose we "define"

$f: \mathbb{Q} \to \mathbb{Z}$

by the rule $f(\frac{m}{n}) = m+n$.

then this "function" isn't one:

$$f(\tfrac{1}{2}) = 1+2 = 3 \neq 6 = 2+4 = f(\tfrac{2}{4})$$

but $\frac{1}{2} = \frac{2}{4}$.

- so f assigns multiple outputs to the same input. what's going on?
- really: there's an implicit equiv relation on fraction representations ($\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots$) and our f is defined in terms of the representative of an equivalence class, not the class itself.
- in general: when given a rule "defining" $f \subseteq A \times B$, to verify f is a function one must show:

① $(\forall a \in A)(\exists b \in B)[(a,b) \in f]$

② if $a = a'$ then $f(a) = f(a')$

Equality of functions:

Q: what does it mean for functions $f: A \to B$ and $g: A \to B$ to be equal?

A: $f = g$ iff they're equal as sets (of ordered pairs), i.e. iff $f \subseteq g$ and $g \subseteq f$.

↪ In practice often easier to use following criterion:

Prop'n: if $f: A \to B$ and $g: A \to B$ are functions then $f = g$ iff

$$(\forall a \in A)\,(f(a) = g(a))$$

pf: you try.

The point: functions can be equal despite being defined by different rules, e.g.: let $A = \{1, 2, 3\}$

define $f: A \to \mathbb{N}$
$g: A \to \mathbb{N}$

by $f(x) = x^3 + 11x$
$g(x) = 6x^2 + 6$.