(More) special notations: conventional to

write $\mathbb{Z}/\equiv_n$ as $\mathbb{Z}/n\mathbb{Z}$.

↳ just like w/ 3, in general we have:

$$\mathbb{Z}/\equiv_n = \mathbb{Z}/n\mathbb{Z} = \{[0]_n, [1]_n, \ldots [n-1]_n\}$$

(2) Let R denote the floor equivalence relation on $\mathbb{R}$: $(x,y) \in R$ iff $\lfloor x \rfloor = \lfloor y \rfloor$

- we checked before: equiv classes all have the form $[n, n+1)$
- indeed for <u>any</u> $x \in [n, n+1)$ we have $[x]_R = [n, n+1)$

e.g. $[0]_R = [.1]_R = [1/2]_R = [.9]_R$

$= [0,1)$

$[1]_R = [1.99]_R = [1.2323\ldots]_R$

$= [1,2)$.

Thus $\mathbb{R}/R = \{[x]_R : x \in \mathbb{R}\}$

$= \{\ldots [-1,0), [0,1), [1,2), \ldots\}$

$= \{\ldots [-1]_R, [0]_R, [1]_R, \ldots\}$

$= \{\ldots [-.6]_R, [.22\ldots]_R, [1.3]_R, \ldots\}$

<u>Notice</u>: in both examples ① + ② the set of equiv. classes forms a partition of the underlying set ($\mathbb{Z}$ in ①, $\mathbb{R}$ in ②).

↳ turns out this is <u>always</u> the case.

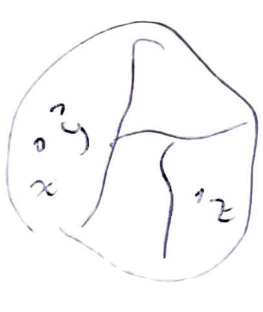
<u>Theorem</u>: If R is an equivalence relation on A, then A/R is a partition of A.

Pf: (HW) For a hint, see 6.7.13 on pg. 449.

<u>Partitions yield equiv. relations</u>:

<u>idea</u>: if P is a partition on A, can define an equiv. relation R on A by "$(x,y) \in R$ iff x and y are in the same piece of the partition"

<u>Pic</u>:



A

so $(x,y) \in R$
but $(x,z) \notin R$.

Let's prove this works:

Thm Sps $\mathbb{P}$ is a partition of A.
Define a relation $R_{\mathbb{P}}$ on A by:

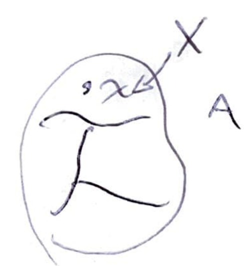
$$(x,y)\in R_{\mathbb{P}} \text{ iff } (\exists X\in\mathbb{P})\,(x\in X \wedge y\in X)$$

Then: $R_{\mathbb{P}}$ is an equiv. relation

Pf (i) (reflexivity)

- Fix $x\in A$.
- Since $\mathbb{P}$ is a partition, we have

$$\bigcup_{X\in\mathbb{P}} X = A$$



- Since $x\in A=\bigcup_{X\in\mathbb{P}} X$
  there is $X\in\mathbb{P}$ s.t. $x\in X$
- hence $x\in X$ also
- hence $(x,x)\in R_{\mathbb{P}}$ ✓

(ii) (symmetry)

- Fix $x,y\in A$ and suppose $(x,y)\in R_{\mathbb{P}}$.
- then $\exists X\in\mathbb{P}$ s.t. $x\in X$ and $y\in X$
- hence $y\in X$ and $x\in X$
- hence $(y,x)\in R_{\mathbb{P}}$ ✓

(iii) (transitivity)

- Fix $x,y,z \in A$ and sps $(x,y) \in R_{\mathbb{P}}$ and $(y,z) \in R_{\mathbb{P}}$.
- then $\exists X \in \mathbb{P}$ s.t. $x \in X$ and $y \in X$ and $\exists Y \in \mathbb{P}$ s.t. $y \in Y$ and $z \in Y$.
- hence $y \in X \cap Y$, in particular $X \cap Y \neq \emptyset$.
- but then it must be $X = Y$, since $\mathbb{P}$ is a partition.
- hence $x \in X^{=Y}$ and $z \in X^{=Y}$
- hence $(x,z) \in R_{\mathbb{P}}$. ✓

<u>Ex's</u> ① Let $\mathbb{P} = \{X, Y, Z\}$ where

$X = \{\dots, -3, 0, 3, 6, \dots\}$

$Y = \{\dots, -2, 1, 4, 7, \dots\}$

$Z = \{\dots, -1, 2, 5, 8, \dots\}$

be our partition of $\mathbb{Z}$ from before.

- Let $R_{\mathbb{P}}$ be the associated equivalence relation: $(x,y) \in R_{\mathbb{P}}$ iff $(\exists S \in \mathbb{P})(x \in S \wedge y \in S)$
- by our theorem, this defines an equiv. relation on $\mathbb{Z}$.

easy to see: this is the same equiv. relation as $\equiv_3$ which we defined before in a different way:

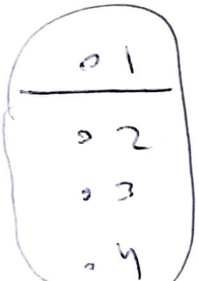
$$x \equiv_3 y \text{ iff } 3 \mid y - x.$$

notice: the equiv. classes of $R_{\mathbb{P}}$ are exactly the pieces of the partition. This always happens, i.e.

$$A/R_{\mathbb{P}} = \mathbb{P}.$$

② - Let $\mathbb{P} = \{\{1\}, \{2,3,4\}\}$ be our partition of $A = [4] = \{1,2,3,4\}$.

- Let $R_{\mathbb{P}}$ be the assoc. equiv. relation.



e.g. $(1,1) \in R_{\mathbb{P}}$
$(2,4) \in R_{\mathbb{P}}$
but $(1,4) \notin R_{\mathbb{P}}$.

- in this case we can actually write out $R_{\mathbb{P}}$ as a set of ordered pairs:

$$R_{\mathbb{P}} = \{(1,1), (2,2), (3,3), (4,4), (2,3), (3,2), (2,4), (4,2), (3,4), (4,3)\}.$$

- no apparent rhyme or reason to this equiv. relation, but still a perfectly good one.

Order Relations

- another common type of binary relation is an order relation
- unlike equiv. relations, these come in several flavors:
  - nonstrict / strict
  - partial / total.

Def'n a relation R on a set A is a (nonstrict) partial order iff R is reflexive, transitive, and antisymmetric.

→ if R is a partial order on A, we say that the pair (A, R) is a partially ordered set or poset.

Ex's ① $\leq$ is a partial order on $\mathbb{R}$.

why: $\forall x, y, z \in \mathbb{R}$ we have:

① $x \leq x$
② $x \leq y \wedge y \leq z \Rightarrow x \leq z$
③ $x \leq y \wedge y \leq x \Rightarrow x = y$

– So we'd say $(\mathbb{R}, \leq)$ is a poset.

② Let A be a fixed set. Then the subset relation $\subseteq$ on $P(A)$ is a partial order.

Why: $\forall X, Y, Z \in P(A)$ we have:

(i) $X \subseteq X$ ✓

(ii) $X \subseteq Y \wedge Y \subseteq Z \Rightarrow X \subseteq Z$ ✓

(iii) $X \subseteq Y \wedge Y \subseteq X \Rightarrow Y = X$ ✓

↳ so $(P(A), \subseteq)$ is a poset.

③ We showed before that the divisibility relation $|$ on $\mathbb{N}$ ($n | m$ iff $(\exists k \in \mathbb{N})\ m = nk$) is reflexive, anti-symmetric and transitive. Hence $|$ is a partial order and $(\mathbb{N}, |)$ is a poset.

Q: is $(\mathbb{Z}, |)$ a poset?

no! remember we already observed that $|$ on $\mathbb{Z}$ is no longer antisymmetric, eg $2 | -2$ and $-2 | 2$ but $2 \neq -2$.

Hence $(\mathbb{Z}, \mid)$ is not a poset

→ these examples of partial orders seem to be of different kinds, and yet — any theorem that can be proved on the basis of only reflexivity, transitivity, and antisymmetry will be true for all posets!

Strict partial orders:

Def'n a relation R on A is irreflexive iff $(\forall x \in A)((x,x) \notin R)$

e.g. $<$ and $\neq$ are irreflexive since we never have $x<x$ or $x \neq x$.

Def'n a relation R on a set A is called a strict partial order if R is (i) irreflexive, (ii) transitive, (iii) antisymmetric

Ex's ① $<$ is a strict partial order on $\mathbb{R}$.

Pf: $\forall x, y, z \in \mathbb{R}$ we have:

(i) $x \nless x$

(ii) $x<y \wedge y<z \Rightarrow x<z$ ✓

(iii) $(x<y \wedge y<x) \Rightarrow x=y$ ✓

because this is never true.

as we can see from this example: combination of irreflexivity and antisymmetry is a bit strange.

<u>Def'n</u>: A relation R on A is called <u>asymmetric</u> iff $(\forall x,y \in A)\ ((x,y)\in R \Rightarrow (y,x)\notin R)$

e.g. $<$ is asymmetric on $\mathbb{R}$ since $x<y \Rightarrow y \nless x$.

<u>Prop'n</u> a relation R on A is asymmetric iff it is both irreflexive and antisymmetric

<u>Pf</u>: ($\Rightarrow$) Sps R is asymmetric and fix $x,y \in A$. (<u>WTS</u>: ① $(x,x)\notin R$ and ② $(x,y)\in R \wedge (y,x)\in R \Rightarrow x=y$)

① is true since if $(x,x)\in R$ this would imply $(x,x)\notin R$ by asymmetry, a contradiction

② is true since by asymmetry $(x,y)\in R \wedge (y,x)\in R$ is always false