Proof of PMI $\Rightarrow$ PSMI

Assume PMI: for every prop'n $P(n)$, if $P(1)$ and $(\forall n \in \mathbb{N})(P(n) \Rightarrow P(n+1))$ hold then $(\forall n \in \mathbb{N}) P(n)$ holds

Want to prove: PSMI: for every prop'n $Q(n)$ if $\overset{(a)}{Q(1)}$ and $\overset{(b)}{(\forall n \in \mathbb{N})}\left((\forall k \in [n]) Q(k) \Rightarrow Q(n+1)\right)$ hold then $(\forall n \in \mathbb{N})\ Q(n)$ holds

- So fix a prop'n $Q(n)$, and assume both (a) and (b) hold
- Let $P(n)$ be the prop'n "$(\forall k \in [n])\ Q(k)$" $\rightarrow$ we'll prove $(\forall n \in \mathbb{N}) P(n)$

(BC) $P(1)$ holds since this is $(\forall k \in [1])\ Q(1)$, which is equiv. to $Q(1)$, which holds by assumption (a).

(IH) Fix $n \in \mathbb{N}$ and assume $P(n)$, i.e. assume $(\forall k \in [n])\ Q(k)$ holds.

(IS) Then by (b) we know $Q(n+1)$ holds. Hence $(\forall k \in [n])\ Q(k)$ and $Q(n+1)$ hold. Hence $(\forall k \in [n+1])\ Q(k)$ holds, i.e. $P(n+1)$ holds

$\Rightarrow$ by (regular) induction (PMI) we have $(\forall n \in \mathbb{N})\, P(n)$ holds, i.e. $(\forall n \in \mathbb{N})(\forall k \in [n])\, Q(k)$ holds. But this implies $(\forall n \in \mathbb{N})\, Q(n)$ holds, as desired ✓.

<u>Proof</u> <u>of</u> <u>WOP $\Rightarrow$ PMI</u>: Assume WOP, and sps $P(n)$ is a variable prop'n. Sps further that

① $P(1)$ holds

② $(\forall n \in \mathbb{N})(P(n) \Rightarrow P(n+1))$ holds.

- Want to prove: $(\forall n \in \mathbb{N})\, P(n)$ holds.
- Let $S = \{n \in \mathbb{N} \mid P(n) \text{ fails}\}$.
- We'll use WOP to prove $S = \emptyset$.
- If $S \neq \emptyset$, then by WOP, S has a least element x.
- We know $x \neq 1$, since $P(1)$ holds
- hence $x = n+1$ for some $n \in \mathbb{N}$.
- Since x is <u>least</u> number for which ~~P~~ P fails, must have $P(n)$ holds.

- but then, by (2), $P(n+1)$ holds, (33)
  i.e. $P(x)$ holds, a contradiction.
- Hence must be $S = \emptyset$.
- Thus gives $(\forall n \in \mathbb{N})\, P(n)$. ✓

↳ PMI, PSMI, WOP are <u>all</u> intuitively obvious principles, and often taken as axioms.

↳ Theorem says: if you assume any one of them, can prove other 2.

Binary relations:

- binary relations are ubiquitous in math
- e.g. we have order relations like:

$$x \leq y$$
$$x < y$$

- the subset relation

$$X \subseteq Y$$

- the divisibility relation

$n \mid m$ ("n divides m")

↳ all assert a relation between two objects (hence "binary")
↳ but what are $\leq, <, \subseteq$ as mathematical objects themselves?
↳ we will define binary relations as sets of ordered pairs.

Def'n Sps A, B are sets. A binary relation on A and B is just a subset $R \subseteq A \times B$.

↳ If $(a,b) \in R$ we say "a is related to b" ⓘ and may sometimes write aRb.

↳ For a relation $R \subseteq A \times B$, we say A is the domain of R, B is the codomain.

↳ Frequently $A = B$, so that $R \subseteq A \times A$. In this case we say: R is a relation on A

Ex's ① Let A = set of Shakespeare's characters
B = set of Shakespeare's plays

- Define a relation $R \subseteq A \times B$ by:
$(a,b) \in R$ iff a appears in b.

- Using set-builder: $R = \{(a,b) \in A \times B \mid a$ appears in $b\}$

- then: (Romeo, "Romeo and Juliet") $\in R$
(Iago, "Othello") $\in R$
(Romeo, "Othello") $\notin R$.

- might also write: Romeo R "Romeo and Juliet"
Romeo $\not R$ "Othello"

② Consider $\leq$ and $<$ as relations on $\mathbb{N}$
Can think of them as sets of pairs.

$\leq$ = $\{(a,b) \in \mathbb{N} \times \mathbb{N} \mid a$ is not greater than $b\}$

$= \{(1,1), (1,2), (5,217), \dots\}$

$< \; = \{(a,b) \in \mathbb{N} \mid a \text{ is strictly less than } b\}$

$= \{(1,2), (17,200), \dots\}$

- we write $1 \le 2$ instead of $(1,2) \in \le$, but these mean the same thing.
- likewise $2 \not< 1$ means $(2,1) \notin <$.
- can visualize $\le$ and $<$ as "triangular" subsets of the grid $\mathbb{N} \times \mathbb{N}$.

$$\begin{array}{ccccc}
(1,4) & (2,4) & (3,4) & (4,4) & \\
(1,3) & (2,3) & (3,3) & (4,3) & \\
(1,2) & (2,2) & (3,2) & (4,2) & \cdots \\
(1,1) & (2,1) & (3,1) & (4,1) & \cdots
\end{array} \qquad \mathbb{N} \times \mathbb{N}.$$

$<$

$\le$

(3) Spr A is a set. Can think of $=$ as a binary relation on A.

$=$ is the set $\{(x,y) \in A \times A \mid x = y\}$

i.e. $\{(x,x) : x \in A\}$.

(4) relations are arbitrary sets of pairs (4)
and need not be defined by some intelligible property, e.g.

- $R = \{(1,1), (2,\pi), (3,\sqrt{2})\} \subseteq \mathbb{R}\times\mathbb{R}$ is a relation on $\mathbb{R}$
- $\emptyset$ is a relation on every set A ("the empty relation")

Properties relations can have:

Def'n: Spse A is a set and $R \subseteq A\times A$ is a relation on A.

1. R is reflexive iff $(\forall x\in A)\,((x,x)\in R)$
2. R is symmetric iff $(\forall x,y\in A)\,((x,y)\in R \Rightarrow (y,x)\in R)$
3. R is transitive iff $(\forall x,y,z\in A)\,(((x,y)\in R \wedge (y,z)\in R) \Rightarrow (x,z)\in R)$
4. R is antisymmetric iff $(\forall x,y\in A)\,((x,y)\in R \wedge (y,x)\in R \Rightarrow x=y)$

Ex's ① on any set A, the equality relation $=$ is reflexive, symmetric, and transitive

(and also antisymmetric, trivially)

↳ relations w/ these three properties are called equivalence relations (more later...)

② $\leq$ (e.g. on $\mathbb{N}$) is reflexive, transitive, and antisymmetric.

why: $(\forall n \in \mathbb{N})\ (n \leq n)$ ✓

$(\forall n, m, \ell \in \mathbb{N})\ (n \leq m \wedge m \leq \ell \Rightarrow n \leq \ell)$ ✓

$(\forall n, m \in \mathbb{N})\ (n \leq m \wedge m \leq n \Rightarrow n = m)$ ✓

but $\leq$ is not symmetric since e.g.

$3 \leq 5$ but $5 \not\leq 3$.

③ $<$ (e.g. on $\mathbb{N}$) is not reflexive or symmetric, but is transitive

(is $<$ antisymmetric? why?)

④ Let $A = \{\text{rock, paper, scissors}\}$

define a relation R on A by:

$(a, b) \in R$ iff a beats b.

is R transitive? Nope:

(scissors, paper) $\in R$
(paper, rock) $\in R$
but (scissors, rock) $\notin R$.

(5) Consider the divisibility relation on $\mathbb{N}$:
$n \mid m$ iff n divides m (i.e. $(\exists k \in \mathbb{N})\, m = nk$)

e.g. $2 \mid 4$ and $2 \mid 12$ but $2 \nmid 7$.

Claim: divisibility $\mid$ on $\mathbb{N}$ is:

(1) reflexive
(2) not symmetric
(3) transitive
(4) antisymmetric

Pf: (1) Fix $n \in \mathbb{N}$. Then $n \mid n$ since $n = n \cdot 1$.
(2) e.g. $2 \mid 4$ but $4 \nmid 2$.
(3) Fix $n, m, l \in \mathbb{N}$ and spse $n \mid m$ and $m \mid l$
i.e. $\exists k_1, k_2 \in \mathbb{N}$ s.t. $m = nk_1$
$l = mk_2$.

but then $l = mk_2 = (nk_1)k_2 = n(k_1 k_2)$
hence $n \mid l$.

(4) Fix $n, m \in \mathbb{N}$ and spse $n | m$ and $m | n$.

then $n = k_1 m$ for some $k_1, k_2 \in \mathbb{N}$.
$m = k_2 n$

$\Rightarrow n = k_1 k_2 n \Rightarrow k_1 k_2 = 1$

$\Rightarrow k_1 = k_2 = 1$

$\Rightarrow n = m$.

(8) Now consider divisibility $|$ on $\underline{\mathbb{Z}}$
i.e. $n | m$ iff $\exists k \in \mathbb{Z}$ $m = nk$.
Then: $|$ remains reflexive, transitive.
still antisymmetric? no: e.g. $2 | -2$
and $-2 | 2$
but $-2 \neq 2$.

<u>Equivalence Relations</u>: <u>Def'n</u> a relation R on a set A is called an <u>equivalence relation</u> iff R is reflexive, symmetric, and transitive.

<u>Ex's</u> (1) On any set A, the equality relation $=$ is an equivalence relation on A: Pf: $\forall x, y, z \in A$ we have:
(i) $x = x$
(ii) $x = y \Rightarrow y = x$
(iii) $x = y \wedge y = z \Rightarrow x = z$

2 Recall: the floor of a real number x, denoted $\lfloor x \rfloor$ is the unique integer n s.t. $n \leq x < n+1$.

e.g. $\lfloor 1.5 \rfloor = 1$

$\lfloor \pi \rfloor = 3$

$\lfloor -2.67 \rfloor = -3$

$\lfloor 5 \rfloor = 5$

Define a relation R on $\mathbb{R}$ by:

$(x,y) \in R$ iff $\lfloor x \rfloor = \lfloor y \rfloor$

i.e. $R = \{(x,y) \in \mathbb{R}^2 \mid \lfloor x \rfloor = \lfloor y \rfloor\}$

Claim R is an equivalence relation

Pf (i) Fix $x \in \mathbb{R}$. Then $\lfloor x \rfloor = \lfloor x \rfloor$. Hence $(x,x) \in R$. Since x was arbitrary, we have $(\forall x \in \mathbb{R})((x,x) \in R)$.

(ii) Fix $x, y \in \mathbb{R}$ and suppose $(x,y) \in R$. Then $\lfloor x \rfloor = \lfloor y \rfloor$. Hence $\lfloor y \rfloor = \lfloor x \rfloor$ (by the symmetry of equality) so $(y,x) \in R$. ✓