Ex: Q: For which $n \in \mathbb{N}$ do we have $n! > 2^n$? Let's see...

n	$n!$	2^n
1	1	2
2	2	4
3	6	8
4	24	16
5	120	32

⋮

Seems like: if $n \geq 4$ then $n! > 2^n$. Let's prove.

Prop'n For every $n \in \mathbb{N}$ w/ $n \geq 4$ we have $n! > 2^n$

(here $n_0 = 4$ and $S = \{n \in \mathbb{Z} \mid n \geq 4\} = \{4, 5, 6, \ldots\}$)

Pf: Let $P(n)$ be the prop'n "$n! > 2^n$"

(BC) $P(4)$ holds since $\underset{24}{4!} > \underset{16}{2^4}$

(IH) Fix $n \in \mathbb{N}$, $n \geq 4$ [NOTE: we fix $n \geq 4$, not $n > 4$]

and assume $P(n)$ holds

i.e. assume $n! > 2^n$.

(IS) Then we have:

$$\begin{aligned} (n+1)! &= n!\,(n+1) \\ &> 2^n\,(n+1) && (\text{by IH}) \\ &> 2^n \cdot 2 && (\text{since } n \geq 4,\ n+1 \geq 5 > 2) \\ &= 2^{n+1} \end{aligned}$$

We've shown $(n+1)! > 2^{n+1}$, i.e. $P(n+1)$ holds.

By induction, we've proved for every $n \geq 4$ we have $n! > 2^n$.

<u>Induction</u> w/ <u>Jumps</u>

– Sometimes we want to prove $P(n)$, not for <u>all</u> n,

but when n is even,

or... when n is odd,

or... when n is a multiple of 3, etc.

↳ can still argue inductively.

<u>Thm</u> Let $P(n)$ be a var prop'n. Fix $n_0 \in \mathbb{Z}$ and $k \in \mathbb{N}$. (n_0 = "starting point", k = "jump")

Let $S = \{n_0, n_0+k, n_0+2k, \dots\}$

If we have

① $P(n_0)$

② $(\forall n \in S)\,(P(n) \Rightarrow P(n+k))$

Then

$(\forall n \in S)\, P(n)$ holds.

---

e.g. if $S = \{2, 4, 6, \dots\} = E$

and we can show

① $P(2)$

② If $P(n)$, then $P(n+2)$

then we've proved $P(n)$ holds $\forall n \in E$.

Ex: Consider the alternating sum of the first n squares

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1} n^2$$

$$= \sum_{k=1}^{n} (-1)^{k-1} k^2$$

Prop'n ① if n is odd we have:

$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k \quad \left(= \frac{n(n+1)}{2} \text{ by before} \right)$$

② if n is even, we have:

$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = -\sum_{k=1}^{n} k \quad \left(= -\frac{n(n+1)}{2} \right)$$

Pf ① here $n_0 = 1$ and jump $= 2$, so that (14)

$S = \{1, 3, 5, \ldots\}$

$P(n)$ is " $\displaystyle\sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k$ "

(BC) if $n = 1$, $\displaystyle\sum_{k=1}^{1} (-1)^{k-1} k^2 = 1^2 = 1 = \sum_{k=1}^{1} k$ ✓

so $P(1)$ holds.

(IH) Fix $n \in \{1, 3, 5, \ldots\}$ and assume $P(n)$,

i.e. assume

$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k.$$

(IS) now consider the $n+2$ sum:

$$\sum_{k=1}^{n+2} (-1)^{k-1} k^2 = \sum_{k=1}^{n} (-1)^k k^2 + (-1)^n (n+1)^2 + (-1)^{n+1} (n+2)^2$$

$$= \sum_{k=1}^{n} (-1)^k k^2 \underbrace{- (n+1)^2 + (n+2)^2}_{\text{since } n \text{ odd}}$$

$$\text{IH} \longrightarrow = \sum_{k=1}^{n} k - (n+1)^2 + (n+2)^2$$

$$= \sum_{k=1}^{n} k + \overbrace{[(n+2) - (n+1)]}^{1}[(n+2) + (n+1)]$$

$$= \sum_{k=1}^{n} k + (n+1) + (n+2)$$

$$= \sum_{k=1}^{n+2} k \qquad \text{so } P(n+2) \text{ holds ✓}$$

By induction we've proved, $\forall n \in \{1,3,5,\dots\}$ (15)

$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k \quad \checkmark$$

<u>Summary</u>: we showed

① $P(1)$ holds

② if $n \in \{1,3,5,\dots\}$ then $P(n) \Rightarrow P(n+2)$

It follows: $P(n)$ holds $\forall n \in \{1,3,5,\dots\}$

② For <u>n even</u>:

(BC) <u>if $n=2$</u>: $\sum_{k=1}^{2} (-1)^{k-1} k^2 = 1^2 - 2^2 = -3$

$$= -(1+2)$$

$$= -\sum_{k=1}^{2} k \quad \checkmark$$

(IH) Fix $n \in \{2,4,6,\dots\}$ and assume:

$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = -\sum_{k=1}^{n} k$$

(IS) Now, consider:

$$\sum_{k=1}^{n+2} (-1)^{k-1} k^2 = \sum_{k=1}^{n} (-1)^{k-1} k^2 + (-1)^n (n+1)^2 + (-1)^{n+1} (n+2)^2$$

$$= \underbrace{-\sum_{k=1}^{n} k}_{IH} + \underbrace{(n+1)^2 - (n+2)^2}_{\text{since } n \text{ even}}$$

$$= -\sum_{k=1}^{n} k + [\overbrace{(n+1) - (n+2)}^{-1}][(n+1)+(n+2)]$$

$$= -\sum_{k=1}^{n} k - [(n+1)+(n+2)]$$

$= -\sum_{k=1}^{n} k - (n+1) - (n+2) = -\sum_{k=1}^{n+2} k$ ✓

by induction the identity holds $\forall n \in \{2, 4, 6, \ldots\}$

<u>Fibonacci sequence</u>: is defined recursively by: $f_0 = 0$, $f_1 = 1$

$f_n = f_{n-2} + f_{n-1}$ for $n \geq 2$

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

$f_0 \; f_1 \; f_2 \; f_3 \; f_4 \; \ldots$

$\rightarrow$ Fib sequence is a playground for inductive proofs.

<u>Prop'n</u> $\forall n \in \mathbb{N}$, we have:

$$\sum_{k=1}^{n} f_k = f_{n+2} - 1$$

(i.e. $f_1 + f_2 + \ldots + f_n = f_{n+2} - 1$)

<u>Pf</u>. (BC) if $n = 1$ we have:

$$\sum_{k=1}^{1} f_k = f_1 = 1 = 2 - 1 = f_3 - 1$$ ✓

(IH) Fix $n \in \mathbb{N}$ and assume

$$\sum_{k=1}^{n} f_k = f_{n+2} - 1$$

(IS) Consider:

$$\sum_{k=1}^{n+1} f_k = \sum_{k=1}^{n} f_k + f_{n+1}$$

$$\overset{IH}{=} (f_{n+2} - 1) + f_{n+1}$$

$$= f_{n+1} + f_{n+2} - 1$$

$$\overset{\text{def'n of } f_{n+3}}{=} f_{n+3} - 1$$

$$= f_{(n+1)+2} - 1$$

by PMI: $\forall n \in \mathbb{N}$ $\sum_{k=1}^{n} f_k = f_{n+2} - 1$ holds.

Prop'n If n is a multiple of 3 (i.e. $n \in \{3, 6, \ldots\}$) then f_n is even.

Pf (BC) if $n = 3$ then $f_n = 2$ which is even.

(IH) Fix $n \in \{3, 6, 9, \ldots\}$ and assume f_n is even.

(IS) Consider f_{n+3}:

$$f_{n+3} = f_{n+2} + f_{n+1}$$
$$= (f_{n+1} + f_n) + f_{n+1}$$
$$= 2f_{n+1} + f_n$$

by the IH, f_n is even. Since $2f_{n+1}$ is even, $2f_{n+1} + f_n$ is too, i.e. f_{n+3} is even. ✓

By induction, $\forall n \in \{3,6,9,\ldots\}$ f_n is even.

<u>Strong induction</u>:

- In certain proofs may need to assume more than $P(n)$ to prove $P(n+1)$
  - e.g. may need to assume $P(n)$ <u>and</u> $P(n-1)$ ... or even $P(n), P(n-1), \ldots,$ and $P(1)$.
  - still a legit induction hypothesis!

<u>Thm</u>" (Principle of strong mathematical induction PSMI)

Sps $P(n)$ is a variable prop'n.

If ① $P(1)$ holds
② $(\forall n \in \mathbb{N}) \left[(\forall k \in [n]) P(k) \Rightarrow P(n+1) \right]$ holds

Then $(\forall n \in \mathbb{N}) P(n)$ holds

"$\forall k \leq n$"

Template for a PSMI proof:

① Prove $P(1)$

② Fix $n \in \mathbb{N}$. Assume $(\forall k \in [n]) P(k)$
(i.e. assume $P(1) \wedge P(2) \wedge \dots \wedge P(n)$)

③ Deduce $P(n+1)$

$\hookrightarrow$ PSMI then gives: $(\forall n \in \mathbb{N}) P(n)$ holds.

Note: - despite name, PSMI seems weaker than PMI, because we have to assume more (namely all of $P(1) \wedge P(2) \wedge \dots \wedge P(n)$ instead of just $P(n)$) to prove $P(n+1)$
- but: we'll later show PMI and PSMI are equivalent (and both equiv to another principle called WOP)

Ex: ① Let s_n be the sequence defined recursively by:

$$s_0 = 1$$
$$s_n = 1 + \sum_{k=0}^{n-1} s_k \quad \text{for } n \geq 1.$$

So e.g.

$$s_1 = 1 + s_0 = 1 + 1 = 2$$
$$s_2 = 1 + s_0 + s_1 = 1 + 1 + 2 = 4$$
$$s_3 = 1 + s_0 + s_1 + s_2 = 1 + 1 + 2 + 4 = 8$$

↳ Looks like $s_n = 2^n$

Let's prove this — we'll need a strong inductive hypothesis.

<u>Prop'n</u> $\forall n \in \mathbb{N} \cup \{0\}$ we have $s_n = 2^n$.

<u>Pf</u>: (BC) if $n = 0$, then $s_0 = 1 = 2^0$ ✓

(Strong IH) Fix $n \in \mathbb{N} \cup \{0\}$ and assume for every $k \in \{0, 1, \ldots, n\}$ we have

$$s_k = 2^k$$

(IS) Now consider:

$$S_{n+1} = 1 + \sum_{k=0}^{n} S_k$$

$$= 1 + \sum_{k=0}^{n} 2^k$$

by strong IH

$$= 1 + \frac{2^{n+1} - 1}{2 - 1}$$

by geometric series formula proved before

Recall: (geo series) if $x \neq 0, 1$ then
$$1 + x + \dots + x^{n-1} = \frac{x^n - 1}{x^n}$$

$$= 2^{n+1} \checkmark$$

$\Rightarrow$ by PSMI, $S_n = 2^n$ for every $n \in \mathbb{N} \cup \{0\}$.

notice: we really needed a strong IH since we need to replace every term in the sum $\sum S_k$ by 2^k, not just the the nth term.

Def'n Given $n \in \mathbb{N}$, $n > 1$, a prime factorization of n is a way of writing n as a product of primes.