Induction :

Q: what happens if we add the first n odd positive integers together?

$$1 = 1 = 1^2$$

$$1 + 3 = 4 = 2^2$$

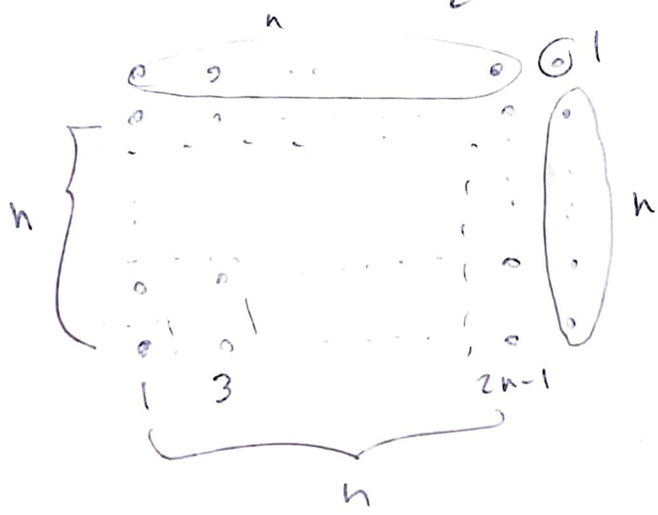
$$1 + 3 + 5 = 9 = 3^2$$

$$1 + 3 + 5 + 7 = 16 = 4^2$$

$$(*) \quad 1 + 3 + 5 + \cdots + (2n-1)^2 \overset{??}{=} n^2$$

Picture

$1 = 1^2$ $\qquad$ $1+3 = 2^2$ $\qquad$ $1+3+5 = 3^2$



$$n^2 + (2n+1) = (n+1)^2$$

- the picture suggests (*) is true for all $n \in \mathbb{N}$. How would we prove it?
- pic also suggests that proof for $n+1$ depends on proof for n.

Theorem: For every $n \in \mathbb{N}$, we have:

$$1+3+5+\cdots+(2n-1) = n^2$$

i.e. $$\sum_{k=1}^{n} 2k-1 = n^2$$

Pf: - Clearly true when $n=1$ since:

$$\sum_{k=1}^{1} 2k-1 = 1 = 1^2$$

- suppose that $n \in \mathbb{N}$ is fixed and we have that the identity holds for n, i.e. assume that

$$\sum_{k=1}^{n} 2k-1 = n^2$$

- now consider the sum for $n+1$:

$$\sum_{k=1}^{n+1} 2k-1 = \underbrace{1+3+\cdots+2n-1}_{\sum_{k=1}^{n} 2k-1} + 2(n+1)-1$$

$$= \sum_{k=1}^{n} 2k-1 \quad + \quad 2n+1$$

$$= \quad n^2 \quad + \quad 2n+1$$

by our assumption

$$= (n+1)^2$$

We've shown:

(a) identity holds for $n=1$

(b) if it holds for a fixed $n \in \mathbb{N}$, then it also holds for $n+1$.

It follows: since identity holds for $n=1$
it holds for $n=2$,
and so also for $n=3$,
and so also for $n=4$,
and so

⋮

and so for all $n \in \mathbb{N}$!

The validity of this kind of argument is called the principle of mathematical induction (PMI) e.g. "$\sum_{1}^{n} 2k-1 = n^2$"

"Theorem" (PMI) Suppose $P(n)$ is a variable proposition. Suppose further that:

1. $P(1)$ holds
2. $(\forall n \in \mathbb{N})\,(P(n) \Rightarrow P(n+1))$ holds

then:

$(\forall n \in \mathbb{N})(P(n))$ holds.

↳ For a "proof" see the book.

↳ we'll take PMI as an axiom (i.e. we'll assume the type of reasoning we used above is valid)

↳ later, we'll show PMI is equivalent to another intuitively obvious principle.

Using PMI to prove: $(\forall n \in \mathbb{N})\, P(n)$

1. (Base case) Verify $P(1)$ directly
2. (Induction hypothesis) Fix $n \in \mathbb{N}$ and assume $P(n)$ holds
3. (Induction Step) using this hypothesis deduce $P(n+1)$.

PMI says: if you can do 1, 2, 3 then $(\forall n \in \mathbb{N})\ P(n)$ holds.

Ex 1 What happens if we sum the first n natural numbers?

$$1+2+\cdots+n = ?$$

First few:

$$1 = 1 = \frac{1\cdot 2}{2}$$

$$1+2 = 3 = \frac{2\cdot 3}{2}$$

$$1+2+3 = 6 = \frac{3\cdot 4}{2}$$

$$1+2+3+4 = 10 = \frac{4\cdot 5}{2}$$

$$\hookrightarrow 1+2+\cdots+n \overset{??}{=} \frac{n(n+1)}{2}$$

Theorem For every $n \in \mathbb{N}$ we have:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Pf: Let $P(n)$ be the prop'n

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

(BC): $P(1)$ is true since

$$\sum_{k=1}^{1} k = 1 = \frac{1\cdot 2}{2} \checkmark$$

(IH): Fix $n \in \mathbb{N}$, and assume $P(n)$, i.e. assume

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

(IS) Now consider:

$$\begin{aligned}
\sum_{k=1}^{n+1} k &= \underbrace{1+2+\cdots+n}+(n+1) \\
&= \sum_{k=1}^{n} k + (n+1) \\
\text{by IH} \quad &= \frac{n(n+1)}{2} + (n+1) \\
&= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\
&= \frac{n(n+1)+2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2} \\
&= \frac{(n+1)((n+1)+1)}{2}
\end{aligned}$$

hence $P(n+1)$ holds.

by PMI, $P(n)$ holds for every $n \in \mathbb{N}$

i.e. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Notice: proof doesn't really give insight into how we might have guessed the formula:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

But: once we have guessed formula, PMI gives us a way of verifying it's really true $\forall n \in \mathbb{N}$.

(2) (Geometric Series) Fix $x \in \mathbb{R}$ with $x \neq 0, 1$. Then for every $n \in \mathbb{N}$ we have:

$$1 + x + x^2 + \cdots x^{n-1} = \frac{x^n - 1}{x - 1}$$

i.e.

$$\underbrace{\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1}}_{\text{call this } P(n)}$$

(BC) $P(1)$ holds since:

$$\sum_{k=0}^{0} x^k = x^0 \underset{\text{since } x \neq 0}{=} 1 \underset{\text{since } x \neq 1}{=} \frac{x^1 - 1}{x - 1}$$

(IH) Fix $n \in \mathbb{N}$ and assume $P(n)$,

i.e. assume $\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x-1}$

(IS) Now consider:

$$\sum_{k=0}^{n} x^k = \sum_{k=0}^{n-1} x^k + x^n$$

$$\text{by IH} \quad = \frac{x^n - 1}{x-1} + x^n$$

$$= \frac{x^n - 1}{x-1} + \frac{x^n(x-1)}{x-1}$$

$$= \frac{\cancel{x^n} - 1 + x^{n+1} - \cancel{x^n}}{x-1}$$

$$= \frac{x^{n+1} - 1}{x-1}$$

$\Rightarrow P(n+1)$ holds.

By PMI, $P(n)$ holds for all $n \in \mathbb{N}$

i.e. $\forall n \in \mathbb{N}$ we have

$$\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x-1} \quad \checkmark$$

3 Prop'n For every $n \in \mathbb{N}$, $7^n - 4^n$ is a multiple of 3.

Pf (BC) If $n=1$, statement holds since $7^1 - 4^1 = 3$.

(IH) Fix $n \in \mathbb{N}$ and assume $\exists k \in \mathbb{N}$ s.t. $7^n - 4^n = 3k$

(IS) Now, observe:

$$7^n = 3k + 4^n \qquad \text{(by IH)}$$

$$\begin{aligned} \Rightarrow 7^{n+1} &= (3k+4^n)7 \\ &= 21k + 7 \cdot 4^n \\ &= 21k + (3+4)4^n \\ &= 21k + 3 \cdot 4^n + 4^{n+1} \end{aligned}$$

$$\begin{aligned} \Rightarrow 7^{n+1} - 4^{n+1} &= 21k + 3 \cdot 4^n \\ &= 3(7k + 4^n) = 3M \end{aligned}$$

where $M = 7k + 4^n$

Hence $7^{n+1} - 4^{n+1}$ is a multiple of 3.

By PMI, $7^n - 4^n$ is a multiple of 3 for every $n \in \mathbb{N}$.

Variants of induction

↳ nothing special about $n=1$ as a base case.

Thm (PMI w/ a different BC)

- Sps $P(n)$ is a var. prop'n and $n_0 \in \mathbb{Z}$ is fixed (possibly negative)
- Let $S = \{n_0, n_0+1, n_0+2, \ldots\} = \{n \in \mathbb{Z} \mid n \geq n_0\}$

If we have

① $P(n_0)$ holds
② $(\forall n \in S)\,(P(n) \Rightarrow P(n+1))$ holds

Then $(\forall n \in S)\, P(n)$ holds

→ can prove theorem using regular PMI (see book)

→ proof template nearly the same as w/ PMI

① (BC) Verify $P(n_0)$
② (IH) Fix $n \in S$. Assume $P(n)$

i.e. $n \in \mathbb{Z}$, $n \geq n_0$

③ (IS) Prove $P(n+1)$