- can also use connectives in def'ns, set-builder notation, etc.
- e.g. if A, B are subsets of a universal set U, can define:

$$A \cap B = \{x \in U \mid (x \in A) \wedge (x \in B)\}$$
$$A \cup B = \{x \in U \mid (x \in A) \vee (x \in B)\}$$
$$\neg A = \{x \in U \mid \neg(x \in A)\}$$

equiv to "$x \notin A$".

- we'll explore connections between connectives and set operations more later.

Implication: - Given statements P, Q the statement $P \Rightarrow Q$ is read "if P, then Q" or "P implies Q"

- $P \Rightarrow Q$ is true iff when P is true, Q is also true.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

notice: - $P \Rightarrow Q$ is always true when P is false (often a confusing point)

- $P \Rightarrow Q$ is only false when P is true and Q is false.

- statements of the form $P \Rightarrow Q$ called <u>conditional</u> <u>statements</u>.

<u>Ex's</u> ① "$(1+1=2) \Rightarrow (1+1+1=3)$" is <u>true</u>

$\uparrow_{T} \qquad\qquad \uparrow_{T}$

② "$(1+1=2) \Rightarrow (1+1+1=4)$" is <u>false</u>

$\uparrow_{T} \qquad\qquad \uparrow_{F}$

③ "$(1+1=2) \Rightarrow (\sqrt{2} \notin \mathbb{N})$" is <u>true</u>

$\uparrow_{T} \qquad\qquad \uparrow_{T}$

<u>even</u> though P and Q in this ex are not apparently related statements.

④ "My name is Sally $\Rightarrow$ My name begins with S" is <u>true</u>

↳ both the premise P and conclusion Q are <u>false</u>, but (by def'n) therefore $P \Rightarrow Q$ is <u>true</u>

↳ illustrates why "false $\Rightarrow$ false" is (T)

⑤ "Tomorrow is Sunday $\Rightarrow$ my name is Garrett" is also <u>true</u>

("false $\Rightarrow$ true" is true)

⑥ $(\exists x \in \mathbb{R})(x^2 = -1) \Rightarrow (1+1=3)$
is true! Automatically since premise is false, even though its unrelated to conclusion.

⑦ can also use $\Rightarrow$ in var. prop'ns e.g.

$$x \geq 2 \Rightarrow x^2 \geq 4$$

is a well-formed var. prop'n, and

$$(\forall x \in \mathbb{R})(x \geq 2 \Rightarrow x^2 \geq 4)$$

is true, because:

$\boxed{\text{For every } x \in \mathbb{R}}$, either $x \geq 2$, in which case $x^2 \geq 4$. Hence $x \geq 2 \Rightarrow x^2 \geq 4$ for such on x, since "true $\Rightarrow$ true" is true;
or $x < 2$, in which case $x \geq 2 \Rightarrow x^2 \geq 4$ is true automatically, since "false $\Rightarrow$ ..." is true.

$\hookrightarrow$ for every $x \in \mathbb{R}$, "$(x \geq 2 \Rightarrow x^2 \geq 4)$" is (T)
i.e. "$(\forall x \in \mathbb{R})(x \geq 2 \Rightarrow x^2 \geq 4)$" is (T) as claimed.

⑧ CTOH: $(\forall x \in \mathbb{R})(x^2 \geq 4 \Rightarrow x \geq 2)$ is false

because: there is a real number x (e.g. $x = -3$) s.t. "$x^2 \geq 4$" is (T)
but "$x \geq 2$" is (F)

i.e. there is $x \in \mathbb{R}$ s.t. "$(x^2 \geq 4) \Rightarrow (x \geq 2)$" is (F)

Equivalence: Given statements P, Q the statement $P \Leftrightarrow Q$
(read: "P if and only if Q" or "P iff Q")
is true iff P, Q have the same truth value.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Ex's ① $(1+1=2) \Leftrightarrow (1+1+1=3)$ is (T)

② $(1+1=3) \Leftrightarrow (1+1+1=4)$ is (T)

③ $(\forall x \in \mathbb{N})(n>0) \Leftrightarrow (1+1=2)$ is (T)

P, Q need not be "related"

④ $(1+1=2) \Leftrightarrow (2+2=5)$ is (F)

- can also use $\Leftrightarrow$ in var. prop'ns, e.g.
$$(x \geq 0) \Leftrightarrow (\exists y \in \mathbb{R})(x=y^2)$$
is a legit var. prop'n, and the statement
$$(\forall x \in \mathbb{R})\left[(x \geq 0) \Leftrightarrow (\exists y \in \mathbb{R})(x=y^2)\right]$$
is true:

why: For any fixed $x \in \mathbb{R}$, the statements "$x \geq 0$" and "$(\exists y \in \mathbb{R})(x=y^2)$" are either both true, or both false.

<u>Def'n</u> Two statements P, Q are said to be <u>logically equivalent</u> iff they have the same truth value, i.e. iff $P \Leftrightarrow Q$ is <u>true</u>.

- e.g. $1+1=2$ and $1+1+1=3$ are logically equiv.
- we're most interested in logically equivalent forms for connected (esp. <u>negated</u>) and quantified statements.

<u>Negating Quantified Statements</u>

- Sps $P(x)$ is a var propn and S a set.
- Consider the negated statements:

① $\neg (\forall x \in S)\, P(x)$

② $\neg (\exists x \in S)\, P(x)$

- <u>Observe</u>: ① is true iff <u>there</u> <u>is</u> an <u>$x \in S$</u> s.t. $P(x)$ is false, i.e. iff $(\exists x \in S) \neg P(x)$ is <u>true</u>

② is true iff <u>for all</u> <u>$x \in S$</u> we have that $P(x)$ is false, i.e. iff $(\forall x \in S) \neg P(x)$ is <u>true</u>.

This shows:

$$\neg(\forall x\in S)\,P(x) \Leftrightarrow (\exists x\in S)\,\neg P(x)$$

is always true (regardless of the prop'n $P(x)$)

i.e. that $\neg(\forall x\in S)\,P(x)$ and $(\exists x\in S)\,\neg P(x)$ are logically equiv

- likewise $\neg(\exists x\in S)\,P(x)$ and $(\forall x\in S)\,\neg P(x)$ are logically equiv.

- these equivalences often useful when trying to prove quantified statements by contradiction.

Ex's: ① $\neg(\forall x\in\mathbb{R})(x\in\mathbb{N})$ "not all reals are naturals"

is equiv. to

$(\exists x\in\mathbb{R})\,\neg(x\in\mathbb{N})$ "there is a real which is not a natural"

(note: we'll often write $\neg(x\in\mathbb{N})$ as $x\notin\mathbb{N}$, $\neg(x=y)$ as $x\neq y$, etc.)

② $\neg(\exists x\in\mathbb{R})(x+1=0)$ "There is no additive inverse for 1 in $\mathbb{R}$"

is equiv to.

$(\forall x\in\mathbb{R})(x+1\neq 0)$ "every real is not an additive inverse for 1"

(in this case, both statements are false)

3. For multiple quantifiers: iterate the process:

$\neg (\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 1)$ — "not every real has a multiplicative inverse"

equiv to: $(\exists x \in \mathbb{R}) \neg (\exists y \in \mathbb{R})(xy = 1)$

" " : $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(xy \neq 1)$ ⇝ "there is a real # w/o a mult. inverse"

(these statements are true: 0 has no mult. inverse)

Negating connected statements:

Prop'n For any statements P, Q, the following logical equivalences hold:

1. $\neg\neg P \iff P$
2. $\neg(P \wedge Q) \iff \neg P \vee \neg Q$
3. $\neg(P \vee Q) \iff \neg P \wedge \neg Q$

"De Morgan's Laws"

PF: to prove, we'll use truth tables

1.

P	$\neg P$	$\neg\neg P$	$\neg\neg P \iff P$
T	F	T	T
F	T	F	T

← always true

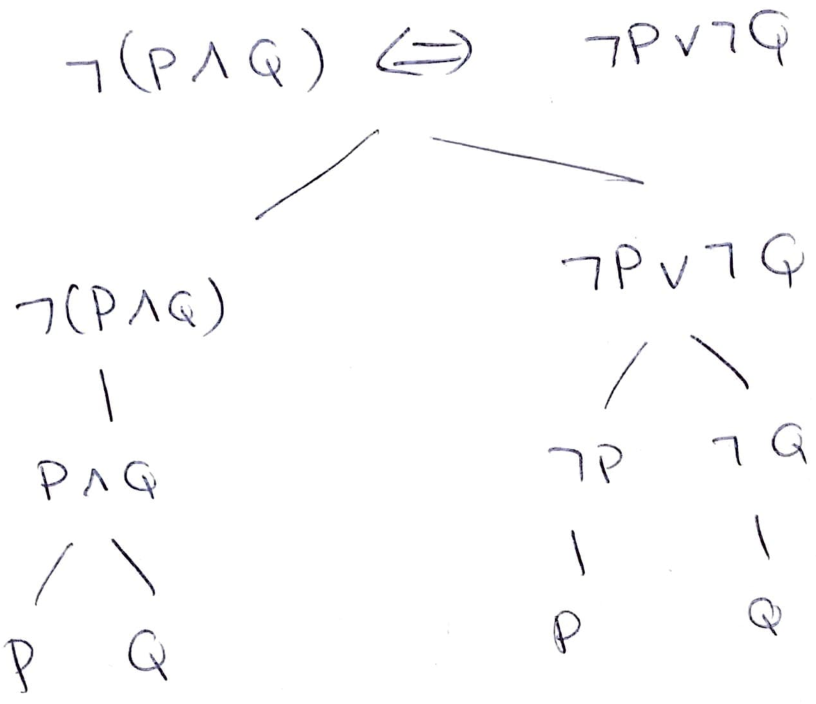
$\neg\neg P \iff P$

$\neg\neg P$ P

$\neg P$

P

(2)



pf of (4)

(3) is similar

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$	$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

always true

Ex's: ① $\neg\neg(1+1=2)$ (both (T))

is equiv to: $1+1=2$

② $\neg((1+1=2) \wedge (1+1=3))$

is equiv. to (both (T))

$(1+1 \neq 2) \vee (1+1 \neq 3)$

③ $\neg((1+1=2) \vee (1+1=3))$ (both (F))

is equiv to: $(1+1 \neq 2) \wedge (1+1 \neq 3)$

④ $(\forall x \in \mathbb{R})\, \neg(x<0 \wedge (\exists y \in \mathbb{R})(y^2=x))$

"is equiv to" $\Leftrightarrow$ $(\forall x \in \mathbb{R})\, [\neg(x<0) \vee \neg(\exists y \in \mathbb{R})(y^2=x)]$

(all true)

Equivalences for $\Rightarrow$: Prop'n For any P, Q the following equivalences hold:

① $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

② $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$

③ $(P \Leftrightarrow Q) \Leftrightarrow (P \Rightarrow Q \wedge Q \Rightarrow P)$

Pf of ① + ②:

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg P \vee Q$	$\neg Q \Rightarrow \neg P$
T	T	T	F	F	T	T
T	F	F	F	T	F	F
F	T	T	T	F	T	T
F	F	T	T	T	T	T

$(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$	$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$
T	T
T	T
T	T
T	T

Pf of ③: you try.

Note: these equivalencies are very useful for proving statements of the form $P \Rightarrow Q$ and $P \Leftrightarrow Q$.

Negating over $\Rightarrow$ and $\Leftrightarrow$ Prop'n:

the following logical equivalences hold.

1. $\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$
2. $\neg(P \Leftrightarrow Q) \Leftrightarrow [(P \wedge \neg Q) \vee (\neg P \wedge Q)]$

Pf: you try.

Note: with these and our previous equivalences, we can now put any negated statement into "positive form"

Def'n A statement P is in positive form iff any negation symbols in P only occur next to substatements that contain no connecting or quantifiers (i.e. negation symbols are "as inside as possible")

↳ our rules above enable us to find, for any P, a logically equiv statement P' in positive form.

Let $E = \{2, 4, 6, \ldots\}$
$O = \{1, 3, 5, \ldots\}$

Ex's ① $(5 \in O) \Rightarrow (6 \in E)$ is equiv

to: $\neg(5 \in O) \vee (6 \in E)$

which we can write:

$(5 \notin O) \vee (6 \in E)$ (T)

(22)

(2) $(\forall x \in \mathbb{N})\ ((x \in O) \Rightarrow (x+1 \in E))$

equiv to:

$(\forall x \in \mathbb{N})\ ((x \notin O) \vee (x+1 \in E))$

also equiv. to:

$(\forall x \in \mathbb{N})\ ((x+1 \notin E) \Rightarrow (x \notin O))$ (T)

(3) Let $P = \{2, 3, 5, 7 \ldots\}$ denote the set of primes. Then:

$(\forall x \in \mathbb{N})\ ((x \in P) \Leftrightarrow (x \in O))$

is equiv. to:

$(\forall x \in \mathbb{N})\ (((x \in P) \Rightarrow (x \in O)) \wedge ((x \in O) \Rightarrow (x \in P)))$

(F).

(4) Consider the following (true) statement: $(\forall x \in \mathbb{R})\,[(x \geq 0) \Leftrightarrow (\exists y \in \mathbb{R})\,(y^2 = x)]$

Let's put its negation in positive form:

$\neg(\forall x \in \mathbb{R})\,[(x \geq 0) \Leftrightarrow (\exists y \in \mathbb{R})\,(y^2 = x)]$

$(\exists x \in \mathbb{R})\ \neg[(x \geq 0) \Leftrightarrow (\exists y \in \mathbb{R})\,(y^2 = x)]$

$\Leftrightarrow$ $(\exists x \in \mathbb{R})\,[((x \geq 0) \wedge \neg(\exists y \in \mathbb{R})\,(y^2 = x))$ ✓

$\Leftrightarrow$ $(\neg(x \geq 0) \wedge (\exists y \in \mathbb{R})\,[y^2 = x))]$

$$\Longleftrightarrow (\exists x\in\mathbb{R})\Big[\big((x\geq 0) \wedge (\forall y\in\mathbb{R})(y^2\neq x)\big) \vee \big((x<0) \wedge (\exists y\in\mathbb{R})(y^2=x)\big)\Big]$$

logically equiv. to orig. negated statement (and False)

More useful equivalency.

Prop'n. The following equivalences hold:

1. $P\wedge(Q\wedge R) \Longleftrightarrow (P\wedge Q)\wedge P$ — Associative Laws
2. $P\vee(Q\vee R) \Longleftrightarrow (P\vee Q)\vee R$
3. $P\wedge(Q\vee R) \Longleftrightarrow (P\wedge Q)\vee(P\wedge R)$ — Distributive Laws.
4. $P\vee(Q\wedge R) \Longleftrightarrow (P\vee Q)\wedge(P\vee R)$

Pf: try the truth tables!

Proving equality of sets using $\Longleftrightarrow$

There is a strong analogy between the logical connectives and the set operations.

From Ch. 3:

connective	operation
$P\wedge Q$	$A\cap B$
$P\vee Q$	$A\cup B$
$P\Rightarrow Q$	$A\subseteq B$
$P\Leftrightarrow Q$	$A=B$
$\neg P$	$\overline{A}$