Chapter 4: Intro to mathematical logic

Goals: - Learn how to write statements more formally (more symbols, fewer words)

- See how the form of a statement suggests the form of its proof.

Recall Def'n (informal) A mathematical statement (or prop'n) is a grammatically correct declarative sentence that is either true or false.

↳ Consists of words and/or symbols

↳ "statement" can be rigorously defined, but need formal logic

↳ "grammatically correct" also has a precise meaning in that context.

Ex's ① Every integer is a real number (T)

② Every real number is an integer (F)

③ There exists an $x \in \mathbb{R}$ s.t. $x \notin \mathbb{Z}$ (T)

4. $1+1=2$ (T)
5. There are infinitely many twin primes (unknown... but either T or F).

<u>Nonex's</u> 1. $\varepsilon\,!\,\pi$ (grammatically incorrect/ meaningless)
2. Shakespeare (not declarative/ no truth value)
3. $x^2+1=2$

↓

a meaningful sequence of symbols asserting an equality... but no truth value <u>unless</u> x <u>is specified</u>.

↳ called a <u>variable</u> <u>proposition</u>: a sentence that becomes a statement once its variables are specified (or quantified over... more later).

↳ will use $P, Q, R, \ldots$ for statements and $P(x), Q(x,y), \ldots$ for var. prop'ns.

<u>e.g.</u> might say: Let P denote "$5^2+1=2$" (F)
Let $Q(x)$ denote "$x^2+1=2$"

Then $Q(5)$ is the statement "$5^2+1=2$" (F)
$Q(1)$ is "$1^2+1=2$" (T)

More var. prop'ns: ① $x^2+1 \leq 0$

② $x \in \mathbb{Z}$ and $x^2 < 39$

③ $z = x + y$

You should indicate when abbreviating a var. prop'n w/ multiple variables.
e.g. could use $Q(x,y,z)$ to denote 3.

Then: $Q(1,2,3)$ is "$1 = 2+3$" (F)
$Q(3,1,2)$ is "$3 = 1+2$" (T)

Quantifiers: other way to turn a var. prop'n into a statement is to quantify over its variables.

ex: "$x^2+1=2$" is a var. prop'n but "there exists $x \in \mathbb{R}$ s.t. $x^2+1=2$" is a statement (T), as is "For every $x \in \mathbb{R}$, we have $x^2+1=2$" (F)

The clauses "There exists $x \in S$ ..."
"For every $x \in S$ ..."
are two types of quantification of the variable x.

- We'll use the symbols

$\forall$ read "for all"

$\exists$ read "there exists"

the "universal quantifier" the "existential quantifier"

- Given a var. prop'n $P(x)$ and a set S, "For all $x \in S$, $P(x)$" and "There exists $x \in S$ such that $P(x)$" are <u>statements</u>

- will denote these by

$(\forall x \in S)\ P(x)$

$(\exists x \in S)\ P(x)$

(Book uses: $\forall x \in S \cdot P(x)$, $\exists x \in S \cdot P(x)$)

respectively.

<u>Ex's</u> ① $(\exists x \in \mathbb{N})\ (x < 5)$

Read "There exists an $x \in \mathbb{N}$ s.t. $x < 5$" (T)

② $(\forall x \in \mathbb{N})\ (x < 5)$

"For every $x \in \mathbb{N}$ we have $x < 5$." (F)

③ $(\forall x \in \mathbb{N})\ (x > 0)$ (T)

④ $(\forall x \in \mathbb{R})\ (x > 0)$ (F)

Multiple Quantifiers:

(5) Consider the var prop'n $x+y \geq 2$.
Then $(\forall y \in \mathbb{N})(x+y \geq 2)$ is still a var. prop'n but now with only one "free variable", namely x. But $(\forall x \in \mathbb{N})(\forall y \in \mathbb{N})(x+y \geq 2)$ is a (T) statement.

Can also write as $(\forall x, y \in \mathbb{N})(x+y \geq 2)$
"For all x and y in $\mathbb{N}$, $x+y \geq 2$"

(6) can also mix $\forall$'s and $\exists$'s, but beware: order of quantifiers is important!

e.g.
$$(\forall x \in \mathbb{N})(\exists y \in \mathbb{R})(y^2 = x)$$
"For every $x \in \mathbb{N}$ there is $y \in \mathbb{R}$ s.t. $y^2 = x$"
i.e. every natural number has a real square root (T)

(7) $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y^2 = x)$
i.e. "every natural number has a square root in $\mathbb{N}$" (F)

↳ what happens if we reverse the order of quantifiers in (6)?

get $(\exists y \in \mathbb{R})(\forall x \in \mathbb{N})(y^2 = x)$

i.e. "there is a real number y s.t. every natural number is equal to y^2"

- perfectly well-written statement, but absurd and definitely false.
- moral: order of quantifiers makes a big diff!

↳ can also have "inside quantifiers"

e.g. (8) $(\forall x \in \mathbb{N})\ (x > 0 \text{ and } (\exists y \in \mathbb{N})(y > x))$

(9) $(\forall x \in \mathbb{R})\ (\text{if } x \geq 0, \text{ then } (\exists y \in \mathbb{R})(y^2 = x))$

are both statements (both (T)).

Note: - we've insisted all quantified variables range over a specified set.

e.g. $(\forall x \in \mathbb{R})(x^2 \geq 0)$ is meaningful but $(\forall x)(x^2 \geq 0)$ is not.

- what if we want to quantify over variables referring to sets?
- e.g. to write "For every set S, we have $\emptyset \subseteq S$"

Symbolically, might try:

$$(\forall S \in (\cdots))\ (\emptyset \subseteq S)$$

set of all sets??

- but the collection of all sets is not a set (Russell's paradox)
- convention: when quantifying over set variables, we'll write sentences verbally:

i.e. "For all sets S ...

" There exist sets A, B ... "

Connectives + Truth Tables

- connectives are symbols used to combine multiple statements into one
- all our connectives will be binary (connect two statements into one) except negation which is unary.
- Truth Tables tell us how truth of a connected statements depends on the truth of the original constituent statements

Conjunction ("and") - conjunction of

two statements P, Q is written
$P \wedge Q$ ("P and Q")

- $P \wedge Q$ is true iff both P, Q are true

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Ex's let P denote:
"$(\forall x \in \mathbb{Z})(x+1 > x)$"

let Q be:
"97 is prime"

let R be:
"$2+2=5$."

- then P, Q are (T) but R is (F).
- hence $P \wedge Q$ is (T)
but $P \wedge R$ and $Q \wedge R$ are both (F).

- written out $P \wedge Q$ is:

$$(\forall x \in \mathbb{Z})(x+1 > x) \wedge (97 \text{ is prime})$$

inserting parentheses can clarify an expression.

Disjunction ("or")

- disjunction of statements P, Q written $P \vee Q$ ("P or Q")
- $P \vee Q$ is true iff at least one of P, Q true.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

e.g. $\underbrace{(\forall x \in \mathbb{R})(x^2 \geq 0)}_{T} \vee \underbrace{(96 \text{ is prime})}_{F}$

is (T), but

$\underbrace{(\forall x \in \mathbb{R})(x \geq 0)}_{F} \vee \underbrace{(96 \text{ is prime})}_{F}$

is (F).

Negation ("not")

- only unary connective
- negation of a statement P written $\neg P$
- $\neg P$ true iff P is false

P	$\neg P$
T	F
F	T

Ex's ① $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y^2 = x)$

is (F), hence

② $\neg(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y^2 = x)$

is (T), hence:

③ $\neg\neg(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y^2 = x)$

is (F) again

④ For any statement P, the statement

$P \vee \neg P$ is (T)

whereas $P \wedge \neg P$ is (F)

e.g. (96 is prime) $\vee$ $\neg$(96 is prime)
is (T)

but (96 is prime) $\wedge$ $\neg$(96 is prime)
is (F).

- Can use connectives in var. prop'ns too:

e.g. Let $P(x,y)$ denote

" $(x > 0) \wedge (y$ is prime$)$ "

then $P(3,5)$ is (T)

while $P(3,6)$ is (F)

also $(\exists x, y \in \mathbb{N})\, P(x,y)$ is (T)

but $(\forall x, y \in \mathbb{N})\, P(x,y)$ is (F)