Then: $\bar{A} = \{4, 5, 6, \ldots\}$
$\bar{E} = \{1, 3, 5, \ldots\} = O$
$\bar{O} = \{2, 4, 6, \ldots\} = E$

Indexing by sets

- It's often useful to take unions / intersections of more than 2 sets
↳ need notation for this.

Ex: For any $i \in \mathbb{N}$, define
$$A_i = \{-i, 0, i\}$$
so, e.g., $A_1 = \{-1, 0, 1\}$, $A_2 = \{-2, 0, 2\}$, etc.

- then $A_1 \cup A_2 = \{-2, -1, 0, 1, 2\}$
$A_1 \cup A_2 \cup A_3 = \{-3, -2, -1, 0, 1, 2, 3\}$

or even $A_1 \cup A_2 \cup \cdots \cup A_{10}$
$= \{-10, -9, \ldots, 8, 9, 10\}$.

↳ we could denote the above union

as
$$\bigcup_{i=1}^{10} A_i$$

- but instead we'll think of the index variable i as "ranging over" the set $[10] = \{1, 2, 3, \ldots, 10\}$ and write

$$\bigcup_{i \in [10]} A_i$$

- in the same way we could write

$$\bigcup_{i \in \{1,2\}} A_i \quad \text{for } A_1 \cup A_2$$

$$\bigcup_{i \in \{1,2,3\}} A_i \quad \text{for } A_1 \cup A_2 \cup A_3$$

More generally:

Def'n: Sps that I is a set (called the index set) s.t. for every $i \in I$ we have defined a set A_i.

we define

$$\bigcup_{i \in I} A_i$$

as the set of el'ts contained in at least one of the A_i's.

i.e. $x \in \bigcup_{i \in I} A_i$

iff there exists an $i \in I$ s.t. $x \in A_i$

We also define:

$$\bigcap_{i \in I} A_i$$

as the set of el'ts contained in <u>every</u> A_i

i.e. $x \in \bigcap_{i \in I} A_i$ iff $x \in A_i$ <u>for every</u> $i \in I$.

<u>Ex's</u>: For $i \in \mathbb{N}$, define $A_i = \{-i, 0, i\}$ as before.

1. Let $I = [10] = \{1, 2, 3, \ldots, 10\}$

Then $\bigcup_{i \in I} A_i = \bigcup_{i \in \{1,2,\ldots,10\}} A_i = \{-10, -9, \ldots, 8, 9, 10\}$

whereas $\bigcap_{i \in I} A_i = \{0\}$.

2. An infinite union:

$$\bigcup_{i \in \mathbb{N}} A_i = \{\ldots, -2, -1, 0, 1, 2, \ldots\} = \mathbb{Z}.$$

3. Let $E = \{2, 4, 6, \ldots\}$

then: $\bigcup_{i \in E} A_i = \{\ldots -2, 0, 2, 4, \ldots\}$.

4. It may be that the indices themselves are sets!

e.g. Let

$$X = \{\{1,2\}, \{1,3\}, \{1,4\}\}$$

What is $\bigcup_{y \in X} y$?

↳ the union of all sets in X!

$$\bigcup_{y \in X} y = \{1,2\} \cup \{1,3\} \cup \{1,4\}$$
$$= \{1,2,3,4\}$$

Similarly $\bigcap_{y \in X} y = \{1,2\} \cap \{1,3\} \cap \{1,4\} = \{1\}$.

(*) See notational insert $\xrightarrow{\text{next}}$ page

next page

Equality of Sets

- a set is determined by its elements: two sets are equal exactly when they have the same elements.

↳ can make this precise w/ a def'n.

Def'n For sets A, B we define

$$A = B \text{ iff } A \subseteq B \text{ and } B \subseteq A$$

i.e. $A = B$ iff whenever $a \in A$, then $a \in B$
and whenever $b \in B$, then $b \in A$

<u>Notation</u> if we've defined A_i for every $i \in I$, we'll write $\{A_i : i \in I\}$ for the <u>set</u> of all sets A_i for $i \in I$.

e.g. if $i = [3] = \{1,2,3\}$ and $A_i = \{-i, 0, i\}$ as above, then

$$\{A_i : i \in [3]\}$$

denotes the set $\{A_1, A_2, A_3\}$

$$= \{\{-1,0,1\}, \{-2,0,2\}, \{-3,0,3\}\}.$$

↳ main relevance of def'n ∪ in proofs:
to prove $A = B$ one proves
(i) $A \subseteq B$ and (ii) $B \subseteq A$
↳ more on this later...

Powersets

- Consider the set $A = \{1,2,3\}$
- Can we list all subsets of A?

Yup:

$$\begin{array}{cccc} & \{1\} & \{1,2\} & \\ \phi & \{2\} & \{1,3\} & \{1,2,3\} \\ & \{3\} & \{2,3\} & \end{array}$$

- the set of all these subsets is called the powerset of A.

$$\{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Def'n Given a set X, the powerset of X, denoted $P(X)$, is the set of all subsets of X, i.e.

$$Y \in P(X) \text{ iff } Y \subseteq X.$$

Ex ① - $\{1, 95, 10^{200}\} \subseteq \mathbb{N}$, hence $\{1, 95, 10^{200}\} \in P(\mathbb{N})$

- Also $E \in P(\mathbb{N})$, $O \in P(\mathbb{N})$, $\mathbb{N} \in P(\mathbb{N})$
- but $\{-1, 0, 1\} \notin P(\mathbb{N})$, since $\{-1, 0, 1\} \not\subseteq \mathbb{N}$.

② Prop'n For any set X we have:

(i) $\emptyset \in P(X)$

(ii) $X \in P(X)$

Pf: we know $\emptyset \subseteq X$ and $X \subseteq X$.

③ Prop'n: For any sets A, B,

if $A \subseteq B$ then $P(A) \subseteq P(B)$

Pf - assume $A \subseteq B$ and fix $X \in P(A)$

- then $X \subseteq A$, by def'n of powerset.
- Then, since $A \subseteq B$, we have $X \subseteq B$ by transitivity of $\subseteq$ (proved before)
- Hence $X \in P(B)$
- Since $X \in P(A)$ was arbitrary, we have $P(A) \subseteq P(B)$ ✓

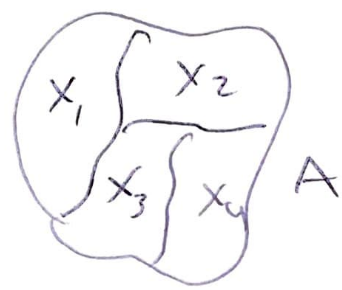
④ What is $P(\emptyset)$?

- only subset of $\emptyset$ is $\emptyset$
- hence $P(\emptyset) = \{\emptyset\}$

Partition

A partition of a set A is a collection of subsets that split A up into disjoint pieces.

Picture:



$\{X_1, X_2, X_3, X_4\}$ is a partition of A into 4 pieces.

Formal Def'n: A partition of a set A is a set P of sets s.t.

(i) for every $X \in P$ we have $X \subseteq A$ and $X \neq \emptyset$.

(ii) for every $X, Y \in P$, if $X \neq Y$ then $X \cap Y = \emptyset$

(iii) $\bigcup_{X \in P} X = A$.

Vocab: (ii) says that the sets $X \in P$ are pairwise disjoint.

Ex's: ① Let $A = \{1,2,3,4,5,6\} = [6]$

Let $S_1 = \{1\}$, $S_2 = \{2,3,6\}$, $S_3 = \{4,5\}$

Then: $P = \{S_1, S_2, S_3\} = \{\{1\}, \{2,3,6\}, \{4,5\}\}$ is a partition of A.

Why: (i) S_1, S_2, S_3 all nonempty subsets of A. ✓

(ii) $S_1 \cap S_2 = S_1 \cap S_3 = S_2 \cap S_3 = \emptyset$ ✓

(iii) $S_1 \cup S_2 \cup S_3 = A$ ✓

However - $\{S_1, S_2\}$ is not a partition of A, since (iii) fails.

- Let $S_4 = \{1,6\}$. Then $\{S_1, S_2, S_3, S_4\}$ is not a partition of A, since (ii) fails (e.g. $S_1 \cap S_4 = \{1\} \neq \emptyset$).

② $\{E, O\}$ is a partition of $\mathbb{N}$.

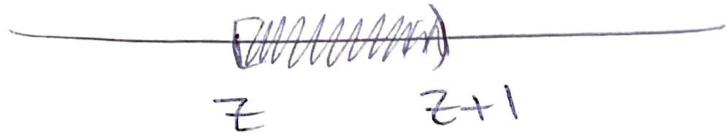
③ - consider $\mathbb{R}$

- for every $z \in \mathbb{Z}$, define

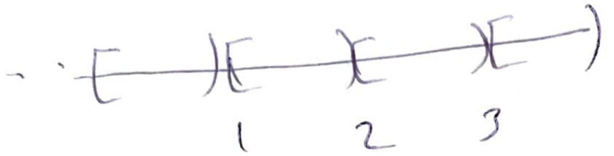
$$S_z = \{x \in \mathbb{R} \mid z \le x < z+1\}$$
$$= [z, z+1)$$

e.g. $S_1 = [1,2)$

S_z



Then $\{S_z : z \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$.
(why?)



(4) OTOH if we define

on the other hand

$$T_z = \{x \in \mathbb{R} \mid z \le x \le z+1\}$$
$$= [z, z+1]$$

then $\{T_z : z \in \mathbb{Z}\}$ is <u>not</u> a partition of $\mathbb{R}$, since e.g. $T_1 \cap T_2$

$$= [1,2] \cap [2,3]$$
$$= \{2\} \neq \emptyset.$$

<u>Cartesian Products</u>: <u>Def'n</u>: Sps A, B are sets. Their <u>Cartesian product</u>, denoted $A \times B$, is the set of all ordered pairs with $a \in A$ and $b \in B$.