2 - finite sets can be defined by just writing all of their elements in brackets

- called roster notation

- e.g. if $A = \{2, 4, 6, \pi\}$

$B = \{♡, *, \pi\}$

Then $\pi \in A$ and $\pi \in B$

while $♡ \in B$ but $♡ \notin A$.

↳ Sets are determined by their elements, order and repetition don't matter.

e.g. if $A = \{1, 2, 3\}$

then $A = \{2, 1, 3\}$

and $A = \{1, 1, 2, 3\}$ as well.

3 - sets can be elements of sets!

- e.g. if $A = \{1, 2\}$ $B = \{3, 4\}$

then $C = \{A, B\} = \{\{1, 2\}, \{3, 4\}\}$

is a legit set.

- different from $D = \{1, 2, 3, 4\}$

(C has 2 el'ts, D has 4)

Some Fundamental Sets

$\mathbb{N} = \{1, 2, 3, \ldots\}$ "natural numbers"
(For us: does not include 0)

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ "integers"

$\mathbb{Q} = \{ \frac{m}{n} \mid m, n \text{ are in } \mathbb{Z} \text{ and } n \neq 0 \}$ "rational numbers"

$\mathbb{R}$ = Set of real numbers

$\mathbb{C}$ = Set of complex numbers
$= \{a + bi \mid a, b \in \mathbb{R}\}$.

So e.g. we have:

$0 \in \mathbb{Z}$ but $0 \notin \mathbb{N}$

$\frac{22}{7} \in \mathbb{Q}$ but $\frac{22}{7} \notin \mathbb{Z}$

$\pi \in \mathbb{R}$ but $\pi \notin \mathbb{Q}$

$\sqrt{-1} \to i \in \mathbb{C}$ but $i \notin \mathbb{R}$.

- the empty set is the unique set with no elements
- denoted $\emptyset$ or $\{\}$
- not the same as $\{\emptyset\}$
  → this set contains a single element, the empty set contains none.

New Sets from old ones.

Set-builder notation: given a set X and a well-defined property P, can form the set Y consisting of all $x \in X$ with property P.

We write $Y = \{x \in X \mid x \text{ has } P\}$
or $Y = \{x \in X \mid P(x)\}$

always need to specify the X from which the x's being drawn from

called "set-builder notation"

Ex's ① can define $E = \{2, 4, 6, \ldots\}$ by:

$E = \{n \in \mathbb{N} \mid n \text{ is a multiple of } 2\}$

or, more symbolically:

"such that"

$E = \{n \in \mathbb{N} \mid \text{there is } k \in \mathbb{N} \text{ s.t. } n = 2k\}$.

② Once E is defined, can use it to define other sets.

e.g. let $O = \{n \in \mathbb{N} \mid \text{there is } k \in E \text{ s.t. } n = k - 1\}$

$= \{1, 3, 5, 7, \ldots\}$

3. The set over which you range is important.

$$\{x \in \mathbb{R} \mid x^2 - 2 = 0\} = \{\sqrt{2}, -\sqrt{2}\}$$

whereas

$$\{x \in \mathbb{Z} \mid x^2 - 2 = 0\} = \emptyset$$

since no integers satisfy $x^2 - 2 = 0$.

<u>More notation</u>: - for a given $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, 2, \ldots, n\}$

- e.g. $[5] = \{1, 2, 3, 4, 5\}$

<u>Subsets</u>:

- a set Y is a <u>subset</u> of X if for every $y \in Y$ we have $y \in X$.
- in this case we write $Y \subseteq X$.
- Y is a <u>proper</u> (or <u>strict</u>) subset of X if $Y \subseteq X$ but $Y \neq X$.
- we (sometimes) write

$$Y \subsetneq X \quad \text{or} \quad Y \subset X$$

to indicate "Y is a proper subset of X"

– whereas $Y \not\subseteq X$ means "Y is not a subset of X"

<u>Ex's</u> ① $\{1,3\} \subseteq \{1,2,3,4\}$

<u>why</u>: $1 \in \{1,2,3,4\}$ and $3 \in \{1,2,3,4\}$

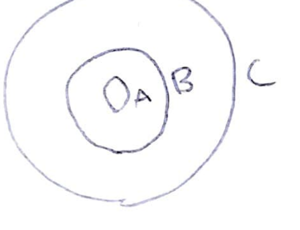
↳ it is a proper subset, so we could write $\{1,3\} \subsetneq \{1,2,3,4\}$
or $\{1,3\} \subset \{1,2,3,4\}$

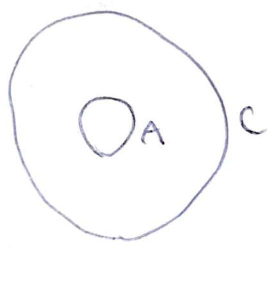
② $\{-5,3\} \not\subseteq \{1,2,3,4\}$

<u>why</u>: $-5 \in \{-5,3\}$ but $-5 \notin \{1,2,3,4\}$

③ $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

<u>Notice</u>: "$\subseteq$" is a <u>transitive</u> relation,
i.e. if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.





let's <u>prove</u> this using the def'n of $\subseteq$.

(ii)

Prop'n 1 For any sets A, B, C, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Pf: - Sps $x \in A$ is a fixed, arbitrary element of A.

- Since $A \subseteq B$ we have $x \in B$, by def'n of $\subseteq$
- Then, since $B \subseteq C$, we have $x \in C$ too, again by def'n of $\subseteq$
- Since $x \in A$ was <u>arbitrary</u>, the same argument would apply to <u>any</u> el't of A.
- Hence <u>every</u> el't of A is an el't of C, i.e. $A \subseteq C$. ✓

More ex's (4) For any set X, we have $X \subseteq X$. **Pf**: fix $x \in X$. Then $x \in X$ too...

(5) Set-builder notation defines a subset, i.e. if $Y = \{x \in X \mid x \text{ has } P\}$ then $Y \subseteq X$.

(6) For any set X we have $\emptyset \subseteq X$.

↳ perhaps unintuitive, but here's why:

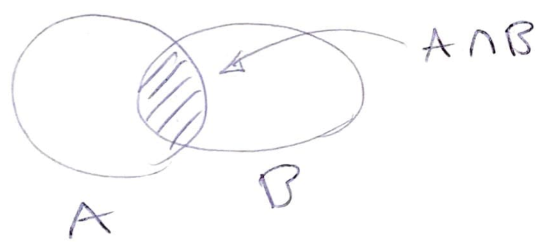
Pf: It is true that
if (i) $x \in \emptyset$
then (ii) $x \in X$

Simply because (i) never holds! → more on this type of reasoning later.

Operations on Sets

Intersections: the intersection of two sets A, B, denoted $A \cap B$, is the set of el'ts belonging to both A and B,

i.e. $x \in A \cap B$ if (and only if) $x \in A$ and $x \in B$.



Ex: if $A = \{1, 2, 3, 4\}$ then: $A \cap B = \{1, 3\}$
$B = \{1, 3, 5\}$ $\quad A \cap C = \{2, 4\}$
$C = \{2, 4, 6\}$ $\quad B \cap C = \emptyset$.

Def'n Two sets are called disjoint iff their intersection is $\emptyset$.

iff = if and only if

ex: B, C above are disjoint.

B (3 1 5) (2 4 6) C

② <u>Prop'n</u> For any sets A, B we have:

(i) $A \cap B \subseteq A$
(ii) $A \cap B \subseteq B$

↳ "obvious" from the picture but let's practice proving from the def'n

<u>Pf</u>: (i) – Fix $x \in A \cap B$
– then $x \in A$ and $x \in B$, by def'n of $\cap$.
– hence in particular $x \in A$.
– Since $x \in A$ was arbitrary, every el't of $A \cap B$ is an el't of A,
i.e. $A \cap B \subseteq A$. ✓

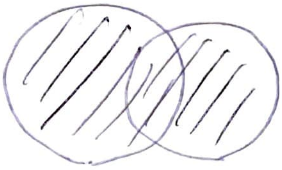
(ii) similar ✓

<u>Unions</u>

<u>Def'n</u>: the <u>union</u> of A and B, denoted $A \cup B$, is the set of el'ts contained in <u>either</u> A <u>or</u> B,

i.e. $x \in A \cup B$
iff $x \in A$ <u>or</u> $x \in B$

Note: "or" here (as in all math) is non exclusive.



$A \cup B$

A B

<u>Ex's</u> ① $\{1,3,5\} \cup \{2,4,6\} = \{1,2,3,4,5,6\}$
$= [6]$.

② if $O = \{1,3,5,\dots\}$ $E = \{2,4,6,\dots\}$
then $O \cup E = \mathbb{N} = \{1,2,3,4,\dots\}$.

③ <u>Prop'n</u>: For any sets A, B
we have: (i) $A \subseteq A \cup B$
(ii) $B \subseteq A \cup B$

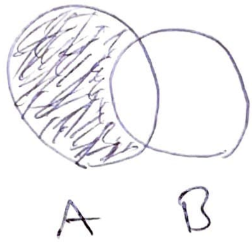
<u>Pf</u>: you try.

<u>Difference</u>

<u>Def'n</u> the <u>difference</u> of two sets A and B, denoted $A-B$, is the set of el'ts in A that are not in B

i.e. $x \in A - B$
iff $x \in A$ and $x \notin B$.

A − B



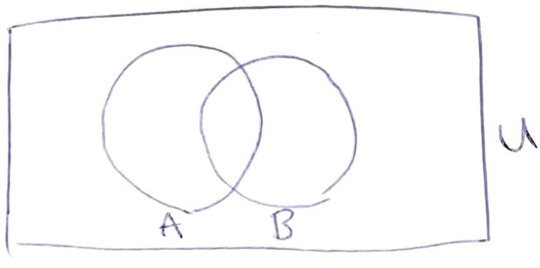
Ex: if $A = \{1,2,3\}$
$B = \{3,4,5\}$

then $A - B = \{1,2\}$ $B - A = \{4,5\}$

notice: difference is not a commutative operation, i.e. $A - B \neq B - A$ in general.

However, $\cap$ and $\cup$ are commutative, i.e. we always have $A \cup B = B \cup A$
$A \cap B = B \cap A$.

Note: in defining $\cap, \cup, -$ it is sometimes convenient to assume our sets A, B are both subsets of a larger set U (called a universal set)



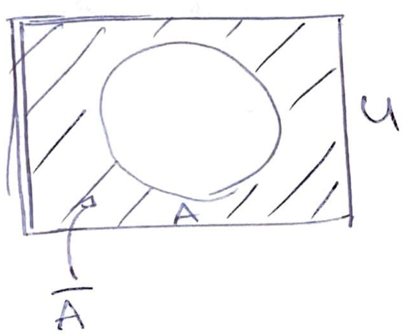
then we can define these operations using set-builder notation:

$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$
$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$
$$A - B = \{x \in U \mid x \in A \text{ and } x \notin B\}.$$
$$= \{x \in A \mid x \notin B\}.$$

Complement

Def'n given a set A, and a universal set U with $A \subseteq U$, the complement of A, denoted $\bar{A}$, is the set of el'ts in U that are not in A

$$\bar{A} = \{x \in U \mid x \notin A\}$$



note: really $\bar{A}$ is just $U - A$.

Ex: ① Sps $U = \mathbb{N}$

$A = \{1, 2, 3\}$
$E = \{2, 4, 6, \ldots\}$
$O = \{1, 3, 5, \ldots\}$