Concepts of Mathematics

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(see Canvas for Zoom link)

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Textbook: Sullivan.pdf (d/l on Canvas)

Grading

HW - 40% (2/week) lowest score dropped.

Exams - 60% (3 exams - 20% each)

Cutoffs - not worse than

90 - 100% = A
80 - 90% = B etc.

Course overview

- Class is an intro to writing proofs
- We'll ~~take~~ a tour: cover some basic set theory, logic, number theory, combinatorics, etc.

... so what is "doing math?"
↳ not just calculating!
- Roughly: it's about investigating mathematical objects (e.g. integers, right triangles, continuous functions) by proving the truth/falsity of mathematical statements about these objects (e.g. "every continuous function is differentiable")
- mathematical objects are described by precise definitions
- e.g. Def'n: A prime number is a positive integer p greater than 1, such that if n is a positive integer that divides p, then either $n=1$ or $n=p$.

Non-defin's: - "A line is a flowing point."
- "A point is a place without extension"
- Emerson.
↳ suggestive, poetical ... but not precise.

- mathematical statements (or propositions) are declarative sentences (concerning math'l objects) that are either true or false (i.e. they have a truth value).

e.g. Prop'n 1: (Euclid) There are infinitely many prime numbers (or, in Euclid's words: "There are more primes than found in any list of primes.")
↳ prop'n 1 is either true or false
either there are infinitely many primes, or not. (in fact: there are)
↳ establishing the truth of a prop'n requires a proof.

- roughly: a proof is a sequence of logical deductions from axioms or previously proved statements whose conclusion is the prop'n in question.
- many methods of proof: one is by contradiction.

Proof of prop'n 1: - Sps ("Suppose") toward a contradiction that prop'n 1 is false, i.e. that there are only finitely many primes

- then we can list them as

  $P_1, P_2, \cdots, P_n$

- Consider the integer

  $$N = P_1 \cdot P_2 \cdots P_n + 1$$

obtained by multiplying all the primes in our list and adding 1.

e.g. maybe 2, 3, 5, 7 are the only primes $\downarrow$ In this case N would be $2 \cdot 3 \cdot 5 \cdot 7 + 1 = 211$

Observe: if we divide N by any of the primes $P_1, \ldots, P_n$ we leave a remainder of 1.

- Hence: N is not divisible by any of the primes $p_1, \dots, p_n$.
- So: <u>either</u> N itself is prime, <u>or</u> there is another prime p not among $p_1, \dots, p_n$ that divides N.

(I'm using here that every integer is either prime, or divisible by some prime)

- either way: there <u>must</u> be another prime not among $p_1, p_2, \dots, p_n$.
  ↳ a contradiction, as we assumed these were <u>all</u> of the primes
- Hence our assumption was <u>false</u>.
- Hence there are infinitely many primes ✓

<u>Sets</u> – a <u>set</u> is a collection of objects (often defined by a common property)

<u>Cantor</u>: "By a 'set' we are to understand any collection into a whole M of definite and separate objects m of our intuition and thought"

- this is an informal def'n (and in fact a contradictory one)
- formal def'n of set beyond scope of course
- our approach: we'll write down several fundamental sets that we'll take for granted, then give formal def'ns of operations that allow us to build new sets from old ones.

- sets are enclosed by curly brackets $\{\dots\}$
- objects in a set are called elements
- $\in$ means "is an element of"
  $\notin$ means "is not an el't of"

Ex's ① Let E denote the set of positive even integers

- we also write:

$$E = \{2, 4, 6, \dots\}$$

Then $12 \in E$, while $1 \notin E$ and $-2 \notin E$.