Homework #7

1. (Contructing the rationals) Define a relation $\sim$ on $\mathbb{Z} \times \mathbb{N}$ such that for any two pairs $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{N}$ we have:

   $$(a, b) \sim (c, d) \Leftrightarrow ad = bc$$

   a. Prove that $\sim$ is an equivalence relation
   b. Determine the set $[(0, 3)]_{\sim}$. That is, define a set using set-builder notation and then prove that this set is $[(0, 3)]_{\sim}$.
   c. Write out three elements of $[(2, 5)]_{\sim}$.
   d. We can naturally identify $(\mathbb{Z} \times \mathbb{N})/\sim$ with one of our standard sets. Which set is this?

2. For a fixed $r \in \mathbb{R}$, the graph of the equation $y = r$ is a horizontal line in the plane $\mathbb{R}^2$. The collection of all such lines is a partition of $\mathbb{R}^2$. Define an equivalence relation on $\mathbb{R}^2$ whose equivalence classes are precisely these horizontal lines.

3. (Modular arithmetic) A key property of the relation of congruence modulo n is that it is preserved by addition and multiplication. In this sense, congruence behaves like equality. For example, from the relation $2 \equiv 5 \pmod 3$ we can, by adding 13 to both sides, deduce $15 \equiv 18 \pmod 3$. And by multiplying both sides by 2 we obtain $4 \equiv 10 \pmod 3$.

   Prove that this works in general. That is, fix $n \in \mathbb{N}$ and prove that for any $x, y, k \in \mathbb{Z}$ we have

   i. if $x \equiv y \pmod n$, then $k + x \equiv k + y \pmod n$
   ii. if $x \equiv y \pmod n$, then $kx \equiv ky \pmod n$

4. Let A be a set and suppose R is a partial order on A (that is, R is a reflexive, transitive, and anti-symmetric relation on A). For $x \in A$ define the *cone of* x, denoted $\langle x \rangle_R$, as follows

   $$\langle x \rangle_R = \{a \in A \mid (a, x) \in R\}$$

   Prove that for all $x, y \in A$, we have $\langle x \rangle_R \subseteq \langle y \rangle_R$ if and only if $(x, y) \in R$.

5. Let A be a set and suppose R is an equivalence relation on A. Prove that set of equivalences classes, A/R, is a partition of A.