Homework #4

1. Consider the following variable propositions:

   Let $P(x)$ be the proposition "$1 \leq x \leq 3$"
   Let $Q(x)$ be the proposition "$(\exists k \in \mathbb{Z})(x = 2k)$"
   Let $R(x)$ be the proposition "$x^2 = 4$"

   For each of the following statements, write the logical negation in positive form. Then decide which claim (the original or the negation) is true (no proof required).

   a.) $(\forall x \in \mathbb{Z})(P(x) \Rightarrow Q(x))$
   b.) $(\exists x \in \mathbb{Z})(R(x) \wedge P(x))$
   c.) $(\forall x \in \mathbb{Z})(R(x) \Rightarrow P(x))$
   d.) $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x \neq y \wedge P(x) \wedge Q(x))$
   e.) $(\forall x \in \mathbb{Z})((P(x) \wedge R(x)) \Leftrightarrow x = 2)$

2. For every $i \in \mathbb{N}$, define a set $A_i \subseteq \mathbb{N}$ such that the indexed family of sets $\{A_i : i \in \mathbb{N}\}$ satisfies *all* of the following properties (recall that "$\subsetneq$" means "is a *strict* subset of"):

   a.) $(\forall n \in \mathbb{N})(\exists i \in \mathbb{N})(n \in A_i)$
   b.) $(\forall i \in \mathbb{N})(\exists n \in \mathbb{N})(n \notin A_i)$
   c.) $(\forall i, m \in \mathbb{N})(\exists n \in \mathbb{N})(n > m \wedge n \in A_i)$
   d.) $(\exists j \in \mathbb{N})(\forall i \in \mathbb{N})(i \neq j \Rightarrow A_j \subsetneq A_i)$

   Then, *prove* that the family you've defined satisfies each of these properties.

3. Consider the following proposition, which asserts that the rational numbers $\mathbb{Q}$ are *dense*:

   Strictly between any two distinct rational numbers lies a third rational number.

   Write this proposition using only logical symbols and the set $\mathbb{Q}$. Then, prove the proposition.

4. Consider the following proposition:

   For all integers n, n is an integer multiple of 3 if and only if $n^2 - 1$ is not a multiple of 3.

   a.) Write out this proposition symbolically, using only logical symbols and the set $\mathbb{Z}$.
   b.) Prove the proposition. (You should be able to prove it using nothing more than the definition of being a multiple of 3, and the fact that every integer has a remainder of 0, 1, or 2 when divided by 3.)