Homework #10

1. Show that there is a bijection $f : [0, 1) \to [0, 1]$ by defining injections $g : [0, 1) \to [0, 1]$ and $h : [0, 1] \to [0, 1)$. Prove that that the functions h and g are injections. (Can you see how to define such a bijection directly?)

2. Fix a prime $p \in \mathbb{N}$ with $p > 3$. Prove that $p^2 \equiv 1 \pmod{24}$.

3. Prove that if p is prime, and for some $a \in \mathbb{Z}$ and $n \in \mathbb{N}$ we have $p \mid a^n$, then $p \mid a$.

4. a) Suppose $n \in \mathbb{N}$ and $n \equiv 3 \pmod{4}$. Show that there is a prime p such that $p \equiv 3 \pmod{4}$ and $p \mid n$.
   b) Prove that there are infinitely many primes p such that $p \equiv 3 \pmod{4}$.

5. Prove the second half of the Fundamental Theorem of Arithmetic, that prime factorizations are unique. That is, prove the following statement. (You may use the fact that prime factorizations exist, since we proved this previously. You may also use Euclid's lemma.)

   For all $n \in \mathbb{N}$, if $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ and $n = q_1^{m_1} q_2^{m_2} \cdots q_l^{m_l}$ are two prime factorizations of n with the primes in each factorization written in ascending order (that is, $p_1 < p_2 < \ldots < p_k$ and $q_1 < q_2 < \ldots < q_l$), then $k = l$ and for all $i \in [k]$ we have $p_i = q_i$ and $n_i = m_i$.