21-127 Concepts of Math

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syllabus w/ schedule

Grading

HW: 40% (2/wk)
Midterm 1: 20%
Midterm 2: 20%
Final: 20%

Textbook: Sullivan (on Canvas)

- begin ch. 3
- wordy but thorough

- HW will be posted on Canvas page as will daily reading assignments + practice problems

Overview

- Class is really an intro to <u>writing</u> <u>proofs</u>

- no single area of focus: cover some basic set theory, logic, number theory, combinatorics, algebra, analysis

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- Roughly: mathematics is the investigation of <u>mathematical objects</u> or <u>concepts</u> (e.g. integers, right triangles, manifolds...) by way of <u>proving</u> the truth or falsity of <u>mathematical statements</u> (e.g. "every diagonal matrix is invertible") about these objects

- Mathematical concepts are described by precise <u>definitions</u>

e.g.

Def'n A prime number is a positive integer p, such that if n is a positive integer that divides p, then either $n = 1$ or $n = p$.

Not def'ns: - "a line is a flowing point"

- "a point is a place without extension"

— Emerson

↳ • suggestive but not precise

- Mathematical statements (or propositions) are declarative sentences (concerning mathematical objects) that are either true or false

e.g.

<u>Prop'n 1</u> There are infinitely many prime numbers.

↳ this is true or false: either there are inf. many primes, or not

↳ to establish the truth of a prop'n requires a <u>proof</u>.

roughly: a sequence of logical deductions from axioms or previously established statements whose conclusion is the prop'n in question

↳ many methods of proof, one is by contradiction

<u>Proof of prop'n 1</u>:

- Suppose toward a contradiction there are only finitely many primes (i.e. suppose prop'n 1 is <u>false</u>)

- then we can list them as
  $p_1, p_2, \ldots, p_n$
- consider the integer

$$N = p_1 p_2 \cdots p_n + 1$$

formed by multiplying the primes and adding 1.

- observe: N is not divisible by any of the primes $p_1, \ldots, p_n$ since division by any of these primes leaves a remainder of 1.
- Hence N must be prime itself or there is another prime not among $p_1, \ldots, p_n$.
- In either case there is another prime not among $p_1, \ldots, p_n$ a contradiction, as we assumed these were all the primes
- Hence our assumption of finitely many primes was false
- Hence there are infinitely many primes.

Sets

A <u>set</u> is a collection of objects (defined by a common property)

↳ Cantor: "By a 'set', we are to understand any collection into a whole M of definite and separate objects m of our intuition or our thought"

↳ this is an informal def'n (and in fact a contradictory one)
↳ formal def'n of <u>set</u> beyond scope of course
↳ we will give examples of several fundamental sets that we "take for granted" and then give <u>rigorous</u> and <u>formal</u> def'ns of several operations that allow us to build new sets from old ones.

- objects in a set are called elements of the set
- sets are enclosed by curly brackets $\{\ldots\}$

- the symbols $\in$ and $\notin$ mean "is an el't of" and "is not an el't of"

Examples

① - let E denote the set of even, positive integers.
- we also write
$$E = \{2, 4, 6, \ldots\}$$
- then $12 \in E$
but $1 \notin E$
and $-2 \notin E$

② - can denote finite sets by simply writing all of their el'ts w/in brackets
- called roster notation
- e.g. let $A = \{2, 4, 6, \pi\}$
$B = \{♡, *, \pi\}$
then $\pi \in A$ and $\pi \in B$
but while $♡ \in B$, we have $♡ \notin A$.

- **Note**: sets are determined by their elements, order and repetition does not matter.

  e.g. if $A = \{1,2,3\}$
  then also $A = \{2,1,3\}$
  and $A = \{1,2,3,1,2\}$

3 - sets can be elements of other sets, e.g.

  let $A = \{1,2\}$
  $B = \{3,4\}$
  and define
  $C = \{A, B\}$
  can also write $C = \{\{1,2\},\{3,4\}\}$
  ↳ not the same as
  the set $D = \{1,2,3,4\}$
  (C has two el'ts, D has four)

Some Fundamental Sets

$\mathbb{N}$ = set of natural numbers
$= \{1, 2, 3, 4, \ldots\}$

(note: our $\mathbb{N}$ does not include 0)

$\mathbb{Z}$ = set of integers
$= \{\ldots -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Q}$ = set of rational numbers
= {numbers of form $\frac{m}{n}$ where $m, n \in \mathbb{Z}$ and $n \neq 0$}

$\mathbb{R}$ = set of real numbers
$\mathbb{C}$ = Set of complex numbers
= {numbers of form $a+bi$, where $a, b \in \mathbb{R}$}

— $\mathbb{N}$ is most fundamental of these, others can be "defined from" $\mathbb{N}$.

e.g.: $0 \in \mathbb{Z}$ $\quad \pi \in \mathbb{R}$
$0 \notin \mathbb{N}$ $\quad \pi \notin \mathbb{Q}$

$i \in \mathbb{C}$
$i \notin \mathbb{R}$

Another important set:

- the *empty set* is the unique set with no el'ts
- denoted { } or $\emptyset$

- *beware*: the set $\{\emptyset\}$ is different than $\emptyset$ (this set contains an el't, the empty set contains none)

New sets from old ones

Set-builder notation

- Given a set X and a well-defined property P, can form a set Y by collecting all x in X with property P

- We write
$$Y = \{x \in X \mid x \text{ has } P\}$$
or
$$Y = \{x \in X \mid P(x)\}$$

- called *set-builder notation*

Ex's

① - Can define set of positive even integers $E = \{2, 4, 6, \ldots\}$:
$$E = \{n \in \mathbb{N} \mid n \text{ is a multiple of } 2\}$$
or, more symbolically:
$$E = \{n \in \mathbb{N} \mid \text{there is a } k \in \mathbb{N} \text{ s.t. } n = 2k\}$$

such that

② - once we've defined E, can use it to define set of positive odd integers

let $O = \{n \in \mathbb{N} \mid \text{there is } k \in E \text{ s.t. } n = k - 1\}$

- then $O = \{1, 3, 5, \ldots\}$

③ the set X over which you range is important

e.g. $\{x \in \mathbb{R} \mid x^2 - 2 = 0\}$ is $\{-\sqrt{2}, \sqrt{2}\}$

but $\{x \in \mathbb{Z} \mid x^2 - 2 = 0\}$ is $\emptyset$

since no integer satisfies $x^2 - 2 = 0$.

Some more notation:

- for a fixed $n \in \mathbb{N}$, $[n]$ refers to the set $\{1, 2, \dots, n\}$
- in set-builder notation: $[n] = \{k \in \mathbb{N} \mid 1 \leq k \leq n\}$

$\boxed{\text{Subsets}}$

Def'n - a set Y is a subset of a set X if for every $y \in Y$ we also have $y \in X$

- ~~notation~~ we write $Y \subseteq X$ for "Y is a subset of X"

Note: - a proper subset is a subset $Y \subseteq X$ such that $Y \neq X$.

- for proper subsets we write $Y \subsetneq X$ or $Y \subset X$

Ex's

① $\{1, 3\} \subseteq \{1, 2, 3, 4\}$

Why: first set contains only 1, 3 and both of these are el'ts of second set.

"not a subset of"

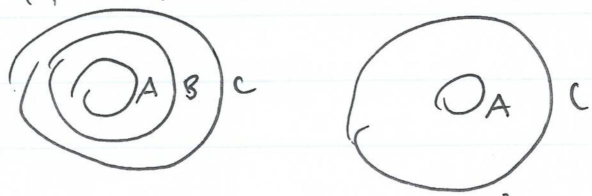
(2) $\{-1,3\} \not\subseteq \{1,2,3,4\}$

why: $-1 \in \{-1,3\}$

but $-1 \notin \{1,2,3,4\}$

(3) $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

↳ Notice: "$\subseteq$" is transitive, i.e. if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$



Let's prove this rigorously

<u>Prop'n</u> For any sets A, B, C, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

<u>Pf</u>: – suppose $x \in A$ is an arbitrary, fixed el't of A

– then since $A \subseteq B$ we have $x \in B$ by def'n of $\subseteq$

– then since $B \subseteq C$ we also have $x \in C$

– Since x was an arbitrary el't of A, every el't of A is an el't of C

– that is $A \subseteq C$.

More ex's

4. $\{x \in \mathbb{R} \mid x - 1 = 0\}$
$\subseteq \{x \in \mathbb{R} \mid x^2 - 1 = 0\}$

<u>Why</u>: first set is $\{1\}$, second is $\{1, -1\}$

5. Set-builder notation defines a subset: if $Y = \{x \in X \mid x \text{ has } P\}$ then $Y \subseteq X$

6. For any X we have $X \subseteq X$
(why: if $x \in X$, then $x \in X$ too...)

7. For any set X we have $\emptyset \subseteq X$.
↳ this is automatic from def'n, but not obvious!
↳ <u>why</u>: it is true that <u>whenever</u> (i) $x \in \emptyset$, <u>then</u> (ii) $x \in X$
simply because (i) never holds!

↳ more on this later ...

Powersets

- Consider the set
  $A = \{1, 2, 3\}$

- Can we list all subsets of A?
- sure:

$\emptyset$ $\{1\}$ $\{2\}$ $\{3\}$ $\{1,2\}$ $\{1,3\}$ $\{2,3\}$ $\{1,2,3\}$ "A"

- We can collect all of these sets together to form a new set:

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

called the powerset of A

Def'n - Given a set X the powerset of X, denoted $P(X)$, is the set of all subsets of X

- that is, $Y \in P(X)$ if and only if $Y \subseteq X$.

Ex's

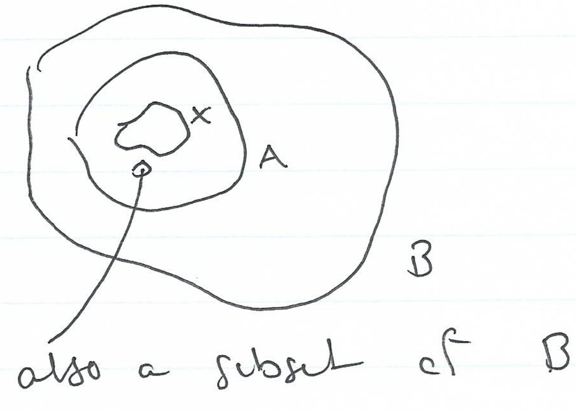
1. - $\{1,2,3\} \subseteq \mathbb{N}$
   - hence $\{1,2,3\} \in P(\mathbb{N})$

   - also $E \in P(\mathbb{N})$
     $O \in P(\mathbb{N})$ since $O, E \subseteq \mathbb{N}$
   - but $\{-1,0,1\} \not\subseteq \mathbb{N}$
     hence $\{-1,0,1\} \notin P(\mathbb{N})$
     but $\{-1,0,1\} \in P(\mathbb{Z})$

2. Prop'n 2 If A and B are sets and $A \subseteq B$, then $P(A) \subseteq P(B)$

Pf: - Sps $X \in P(A)$ is an arbitrary but fixed el't of $P(A)$
"suppose"
- then by def'n of $P(A)$ we have $X \subseteq A$
- Since $X \subseteq A$ and $A \subseteq B$, by Prop'n 1 we have $X \subseteq B$
- hence $X \in P(B)$ by def'n
- since $X \in P(A)$ was arbitrary, we know for every $X \in P(A)$ we have $X \in P(B)$
- hence $P(A) \subseteq P(B)$

Picture



3 - What is $P(\emptyset)$?
- well: only subset of $\emptyset$ is $\emptyset$ itself!
- hence $P(\emptyset) = \{\emptyset\}$

Equality of Sets

- We said before: a set is determined by its elements, i.e. two sets are the same when they have the same elements.

- Can make this into a formal def'n using $\subseteq$.

Def'n Given sets A, B we say $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

↳ unraveling this: says $A = B$

iff whenever $a \in A$ then $a \in B$ and whenever $b \in B$ then $b \in A$

"if and only if"

Ex's

① if $A = \{1, 2, 3\}$ and $B = \{2, 1, 3\}$ then $A = B$.

↳ main import of def'n is in proofs: to prove two sets A, B are equal, show ① $A \subseteq B$
② $B \subseteq A$

↳ such pf's are called double containment pf's.

Combining Sets with Operations

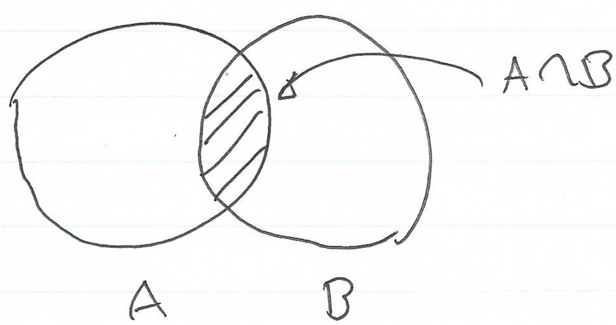
Intersections

Def'n Given sets A, B, the intersection of A and B, denoted $A \cap B$, is the set of el'ts belonging to both A and B,

i.e.

$$x \in A \cap B \quad \text{iff} \quad x \in A \text{ and } x \in B$$

Picture

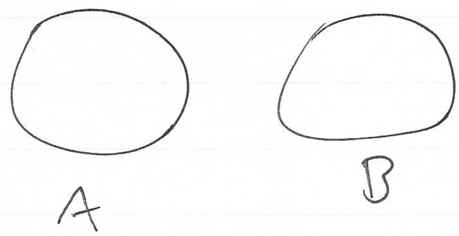


Ex's

1. Let $A = \{1, 2, 3, 4\}$
$B = \{1, 3, 5\}$
$C = \{2, 4, 6\}$

then $A \cap B = \{1, 3\}$
$A \cap C = \{2, 4\}$
$B \cap C = \emptyset$

<u>Def'n</u> if A, B are sets and $A \cap B = \emptyset$ we say A and B are <u>disjoint</u>.



disjoint sets

② <u>Prop'n</u> For any sets A, B we have ~~$A \subseteq A \cap B$ and $B \subseteq$~~ (i) $A \cap B \subseteq A$

(ii) $A \cap B \subseteq B$

<u>Pf</u>: (i) - if $x \in A \cap B$ is arbitrary + fixed then by def'n $x \in A$ and $x \in B$

- hence $x \in A$
- hence for every $x \in A \cap B$ we have $x \in A$
- hence $A \cap B \subseteq A$ ✓

Similar pf for (ii) ✓

<u>Union</u>

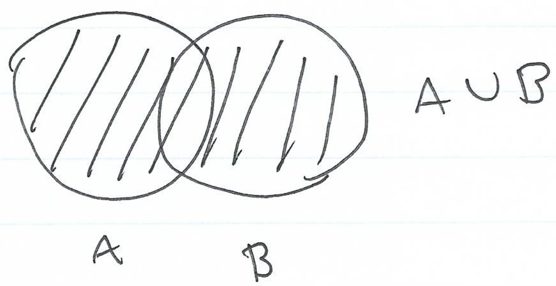
<u>Def'n</u> if A, B are sets, the union of A and B, denoted $A \cup B$, is the set of el'ts contained in either A or B,

i.e.

$x \in A \cup B$ iff $x \in A$ <u>or</u> $x \in B$

- Note: "or" here (as in all math) is non-exclusive:
- i.e. $x \in A \cup B$ iff $x \in A$ or $x \in B$ (or both)

<u>Pic</u>



<u>Ex's</u>

① $\{1,3,5\} \cup \{2,4,5\}$
$= \{1,2,3,4,5\}$

② if $E = \{2,4,6,\ldots\}$
$O = \{1,3,5,\ldots\}$
then $E \cup O = \mathbb{N}$

③ <u>Prop'n</u> For any sets A, B we have
(i) $A \subseteq A \cup B$
(ii) $B \subseteq A \cup B$

<u>Pf</u>: try it yourself . . .

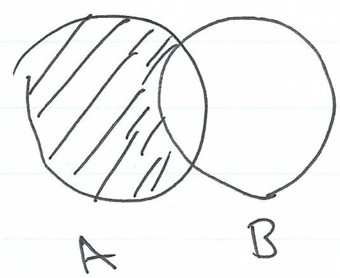
Difference

Def'n if A, B are sets, the difference of A and B, denoted $A-B$ is set of el'ts in A that are not in B

i.e. $x \in A-B$ iff $x \in A$ and $x \notin B$

Pic

A − B



ex's

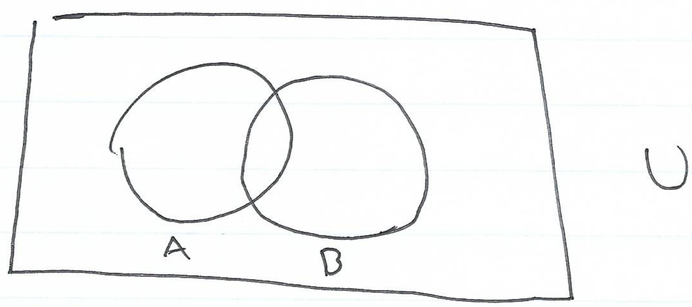
① if $A = \{1,2,3\}$
$B = \{3,4,5\}$
then $A - B = \{1,2\}$
$B - A = \{4,5\}$

↳ shows difference is not symmetric: $A-B \neq B-A$ in general
↳ however $\cup$ and $\cap$ are symmetric:
$A \cup B = B \cup A$ and $A \cap B = B \cap A$
for any A, B.

② if $E = \{2, 4, 6, \ldots\}$
$O = \{1, 3, 5, \ldots\}$

then $\mathbb{N} - E = O$
$\mathbb{N} - O = E$

<u>Note</u>: - in defining $\cap, \cup, -$ it is sometimes convenient to assume our sets A, B are subsets of another set U (called a universal set)



- then we can define these operations using set builder notation:

$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$
$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$
$$A - B = \{x \in U \mid x \in A \text{ and } x \notin B\}$$

Complement

Def'n Given a set A and a universal set U s.t. $A \subseteq U$, the complement of A, denoted $\bar{A}$, is the set of el'ts in U that are not in A:

$$\bar{A} = \{x \in U \mid x \notin A\}$$

Pic:

note: - $\bar{A}$ only defined relative to U
- really: $\bar{A}$ is just $U - A$

Examples

(1) if $U = \mathbb{N}$
$A = \{1, 2, 3\} = [3]$
$E = \{2, 4, 6, \ldots\}$
$O = \{1, 3, 5, \ldots\}$

then $\bar{A} = \{4, 6, 6, \ldots\}$
$\bar{E} = \{1, 3, 5, \ldots\} = O$
$\bar{O} = \{2, 4, 6, \ldots\} = E$

Indexing by Sets

↳ $\cup$ and $\cap$ allow us to combine two sets in certain ways
↳ often useful to take unions/intersections of more than two sets
↳ need notation for indexing larger collections of sets

Motivating ex

- For any $i \in \mathbb{N}$, define $A_i = \{-i, 0, i\}$
- so $A_1 = \{-1, 0, 1\}$
  $A_2 = \{-2, 0, 2\}$
  $A_3 = \{-3, 0, 3\}$ etc.

- then $A_1 \cup A_2 = \{-2, -1, 0, 1, 2\}$
  $A_1 \cup A_2 \cup A_3 = \{-3, -2, -1, 0, 1, 2, 3\}$

- or even

$$A_1 \cup A_2 \cup \dots \cup A_{10}$$
$$= \{-10, -9, \dots, -1, 0, 1, \dots, 9, 10\}$$

- we might write the above union more formally as:

$$\bigcup_{i=1}^{10} A_i$$

- alternately, instead of thinking of the index variable i as "clicking up" from 1 to 10 we might think of it as ranging over the set $[10] = \{1, 2, \ldots, 10\}$ and write union as

$$\bigcup_{i \in [10]} A_i$$

- this idea of letting an index var. range over a set is useful.

Def'n Sps I is a set (called an index set) s.t. for every $i \in I$ we have defined a set A_i

Then we define

$\bigcup_{i \in I} A_i$ as the set of el'ts contained in at least one of the A_i

i.e. $x \in \bigcup_{i \in I} A_i$

iff there is an $i \in I$ such that $x \in A_i$

We also define $\bigcap_{i \in I} A_i$ as the set of el'ts contained in every A_i

i.e. $x \in \bigcap_{i \in I} A_i$ iff $x \in A_i$ for every $i \in I$

Examples

For $i \in \mathbb{N}$, define $A_i = \{-i, 0, i\}$ as before

① let $I = \{1, 2, \ldots, 10\}$
then

$$\bigcup_{i \in I} A_i = \bigcup_{i \in \{1, \ldots, 10\}} A_i$$

$$= A_1 \cup A_2 \cup \cdots \cup A_{10}$$

$$= \{-10, -9, \ldots, -1, 0, 1, \ldots, 9, 10\}$$

② An infinite union:

$$\bigcup_{i \in N} A_i = A_1 \cup A_2 \cup \cdots$$

$$= \{\ldots, -1, 0, 1, \ldots\}$$

$$= \mathbb{Z}.$$

③ Let $E = \{2, 4, 6, \ldots\}$

then $\bigcup_{i \in E} A_i = A_2 \cup A_4 \cup A_6 \cup \cdots$

$$= \{\ldots, -4, -2, 0, 2, 4, \ldots\}$$

④ OTOH:

$$\bigcap_{i \in [10]} A_i = A_1 \cap A_2 \cap \cdots \cap A_{10}$$

$$= \{-1, 0, 1\} \cap \{-2, 0, 2\} \cap \cdots \cap \{-10, 0, 10\}$$

$$= \{0\}$$

Since 0 is only el't contained in <u>every</u> A_i

↳ in fact, $A_1 \cap A_2 = \{0\}$ already.

(5) - Let $J = \{1,2,3\}$ and for every $j \in J$ define

$$B_j = \{j-2, j-1, j, j+1, j+2\}$$

- so: $B_1 = \{-1, 0, 1, 2, 3\}$
  $B_2 = \{0, 1, 2, 3, 4\}$
  $B_3 = \{1, 2, 3, 4, 5\}$

- then: $\bigcup_{j \in J} B_j = \{-1, 0, 1, 2, 3, 4, 5\}$

whereas $\bigcap_{j \in J} B_j = \{1, 2, 3\}$

(6) - It may be the indices are themselves sets
- e.g. let
  $X = \{\{1,2\}, \{1,3\}, \{1,4\}\}$

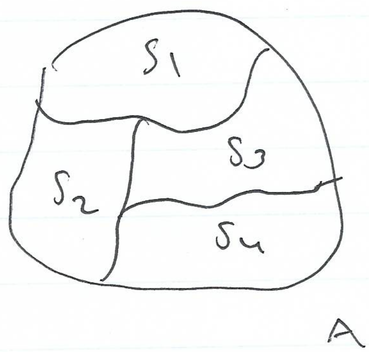
then $\bigcup_{y \in X} y = \{1,2\} \cup \{1,3\} \cup \{1,4\}$
$= \{1,2,3,4\}$

and $\bigcap_{y \in X} y = \{1,2\} \cap \{1,3\} \cap \{1,4\}$
$= \{1\}$

Partitions

A partition of a set A is a collection of subsets of A that split up A into disjoint pieces

Pic



Formal def'n: Let A be a set. A partition of A is a collection of nonempty sets S_i, $i \in I$ such that

(i) for every $i \in I$, $S_i \subseteq A$
(ii) if $i \neq j$ then $S_i \cap S_j = \emptyset$
(iii) $\bigcup_{i \in I} S_i = A$

↳ vocab: (ii) says the sets S_i are "pairwise disjoint"

Ex's

① Let $A = \{1,2,3,4,5,6\}$

Let $S_1 = \{1,2\}$

$S_2 = \{3,4\}$

$S_3 = \{5,6\}$

then $\{S_1, S_2, S_3\} = \{\{1,2\},\{3,4\},\{5,6\}\}$

is a partition of A

Check: (i) $S_1, S_2, S_3 \subseteq A$

(ii) $S_1 \cap S_2 = \emptyset$

$S_1 \cap S_3 = \emptyset$

$S_2 \cap S_3 = \emptyset$

(iii) $\bigcup_{i \in \{1,2,3\}} S_i = \{1,2,3,4,5,6\}$

$= A$

However: - $\{S_1, S_2\}$ is not a partition of A (condition (iii) fails)

- let $S_4 = \{1,6\}$

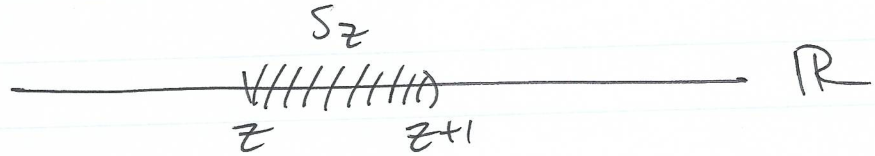
- then $\{S_1, S_2, S_3, S_4\}$ is not a partition of A since e.g. $S_1 \cap S_4 = \{1\} \neq \emptyset$

2. Sometimes we drop the indices:
e.g. $\{E, O\}$ is a partition of $\mathbb{N}$.

3. - Consider the set $\mathbb{R}$.
- For every $z \in \mathbb{Z}$, define

$$S_z = \{x \in \mathbb{R} \mid z \le x < z+1\}$$

$$= [z, z+1)$$



Can you see why $\{S_z : z \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$?

4. OTOH: if we define
$$T_z = \{x \in \mathbb{R} \mid z \le x \le z+1\}$$
$$= [z, z+1]$$

then $\{T_z : z \in \mathbb{Z}\}$ is <u>not</u> a partition of $\mathbb{R}$ since these sets are not pairwise disjoint
e.g. $T_1 \cap T_2 = [1,2] \cap [2,3]$
$= \{2\} \neq \emptyset$.

Cartesian Products

Familiar example

- $\mathbb{R} \times \mathbb{R}$ = set of ordered pairs of real numbers

$$= \{ (a,b) \mid a \in \mathbb{R} \text{ and } b \in \mathbb{R} \}$$

technically an illegal use of set-builder notation, but you get idea.

More generally:

Def'n if A, B are sets, the Cartesian product of A and B, denoted $A \times B$ is the set of pairs (a,b) with $a \in A$ and $b \in B$

$$A \times B = \{ (a,b) \mid a \in A, b \in B \}$$

Note: - order is important here!

- first coord. from A, second from B
- could have $A \neq B$

↳ we sometimes denote $A \times A$ by A^2.

Ex's

(1) if $A = \{1, 2, 3\}$
$B = \{*, \heartsuit\}$
then $A \times B = \{(1,*), (2,*), (3,*), (1,\heartsuit), (2,\heartsuit), (3,\heartsuit)\}$

(2) - $(1, \pi^2)$ and $(2, \sqrt{2})$ are elements of $\mathbb{N} \times \mathbb{R}$

- $(\pi, \sqrt{2}) \notin \mathbb{N} \times \mathbb{R}$ since $\pi \notin \mathbb{N}$

- but $(\pi, \sqrt{2}) \in \mathbb{R} \times \mathbb{R}$

(3) Triples: Given sets A, B, C can define $A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$

e.g. $(1, \pi, 2+i) \in \mathbb{N} \times \mathbb{R} \times \mathbb{C}$

- $A \times B \times C$ is not the same as $(A \times B) \times C$: el'ts of first set are (a, b, c), el'ts of second are $((a, b), c)$

- but these sets are essentially the same.

Proofs with sets

Proving $A \subseteq B$

Strategy: 1. Fix an arbitrary $a \in A$
2. Show $a \in B$
3. conclude, since a was arbitrary, that for every $x \in A$ we have $x \in B$, i.e. $A \subseteq B$

ex. ① Prop'n Let A, B, X be sets. If $X \subseteq A$ and $X \subseteq B$ then $X \subseteq A \cap B$

Pf: - Fix an arbitrary $x \in X$
- Since $X \subseteq A$, we have $x \in A$
- Since $X \subseteq B$, we also have $x \in B$
- hence $x \in A \cap B$
- Since x was arbitrary, we have shown for every $x \in X$ we have $x \in A \cap B$.
- Hence $X \subseteq A \cap B$. ✓

② Prop'n Let A, B be sets. Then $P(A) \cap P(B) \subseteq P(A \cap B)$

Pf: - Fix an arbitrary $X \in P(A) \cap P(B)$
- Then $X \in P(A)$ and $X \in P(B)$, by defn of $\cap$

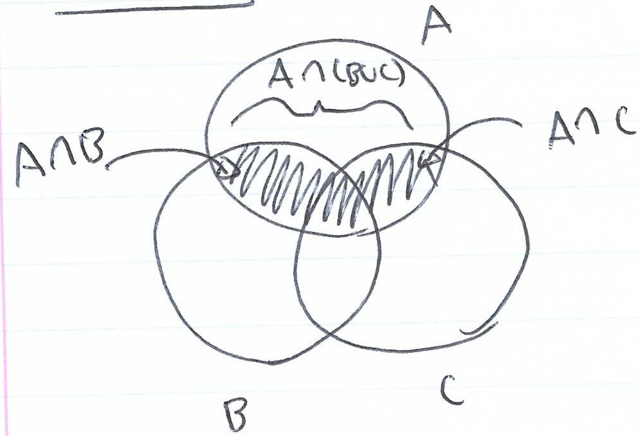
- Hence $X \subseteq A$ and by def'n of power
- Hence, by our previous pr
$X \subseteq A \cap B$
- Therefore $X \in P(A \cap B)$
- Since X was arbitrary, we have
$P(A) \cap P(B) \subseteq P(A \cap B)$

Proving $A = B$

__Strategy__:
1. Show $A \subseteq B$
2. Show $B \subseteq A$
3. Conclude $A = B$

__ex's__ ① __Prop'n__ Let A, B, C be sets. Then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

__Picture__:



Pf: ($\subseteq$) - Fix $x \in A \cap (B \cup C)$
- then (i) $x \in A$ and
(ii) $x \in B \cup C$
- Case 1: $x \in B$. In this case, since $x \in A$ we have $x \in A \cap B$.
- Case 2: $x \notin B$. In this case we must have $x \in C$, since from (ii) we know that either $x \in B$ or $x \in C$. Hence, since $x \in A$ as well we have $x \in A \cap C$.
- We have shown that either $x \in A \cap B$ or $x \in A \cap C$, that is $x \in (A \cap B) \cup (A \cap C)$
- Since x was arbitrary we have $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ ✓

($\supseteq$) - Now suppose $x \in (A \cap B) \cup (A \cap C)$
- Case 1: $x \in A \cap B$
  - in this case, $x \in A$ and $x \in B$
  - hence, $x \in A$ and ($x \in B$ or $x \in C$)
  - that is, $x \in A \cap (B \cup C)$
- Case 2: $x \notin A \cap B$.
  - in this case we must have $x \in A \cap C$
  - hence, $x \in A$ and $x \in C$
  - hence, $x \in A$ and ($x \in B$ or $x \in C$)
  - that is, $x \in A \cap (B \cup C)$

- Thus in all cases $x \in A \cap (B \cup C)$
- Since x was arbitrary we have $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ ✓

- We have shown $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$
- Thus these sets are equal. ✓

Counterexamples

- To show a statement is false, sufficient to provide a single counterexample

Ex Consider the following statement: For any sets A, B, C, if $A \subseteq B \cup C$ then either $A \subseteq B$ or $A \subseteq C$.

↳ this statement is false

↳ consider $A = \{2,3\}$
$B = \{1,2\}$
$C = \{3,4\}$

- Then $B \cup C = \{1,2,3,4\}$
so $A \subseteq B \cup C$
but neither $A \subseteq B$ nor $A \subseteq C$
- So statement is false

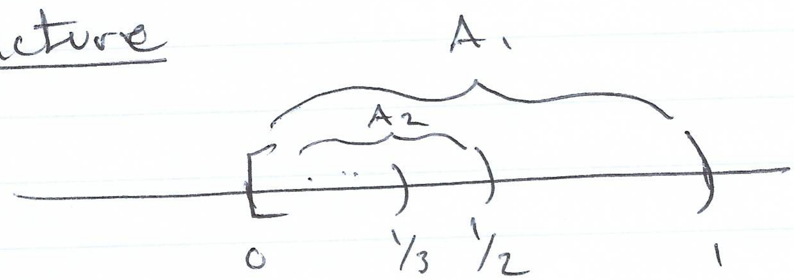
Using contradiction

idea: to prove a statement S is true, you can
1. Assume S is false
2. Show this assumption contradicts your hypotheses (or a previously established statement)
3. Conclude S is true.

ex: For every $n \in \mathbb{N}$, define $A_n = \{x \in \mathbb{R} \mid 0 \leq x < 1/n\}$.
Then

$$\bigcap_{n \in \mathbb{N}} A_n = \{0\}$$

Picture



Pf: We prove $\supseteq$ first, since this is easier.

($\supseteq$) – To show $\{0\} \subseteq \bigcap_{n\in\mathbb{N}} A_n$, it is sufficient to show $0 \in \bigcap_{n\in\mathbb{N}} A_n$.

– By def'n, $0 \in \bigcap_{n\in\mathbb{N}} A_n$ iff $0 \in A_n$ for every $n \in \mathbb{N}$.

– Fix $n \in \mathbb{N}$

– Then $0 \in A_n$ since $A_n = \{x \in \mathbb{R} \mid 0 \le x < 1/n\}$ and of course $0 \le 0 < 1/n$

– Since n was arbitrary, $0 \in A_n$ for every n.

– That is $0 \in \bigcap_{n\in\mathbb{N}} A_n$. ✓

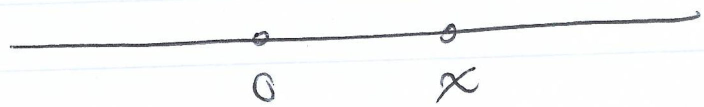
($\subseteq$) – We now show $\bigcap_{n\in\mathbb{N}} A_n \subseteq \{0\}$

toward a contradiction

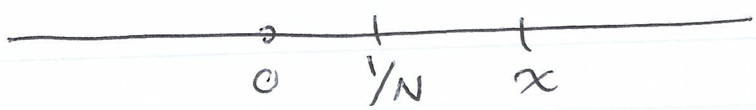
– Suppose not. Then there must be some $x \in \bigcap_{n\in\mathbb{N}} A_n$ s.t. $x \notin \{0\}$

– hence $x \neq 0$

– hence we must have $x > 0$, since for every n and every $y \in A_n$ we know $y \ge 0$ and we know $x \neq 0$.



- (the trick): Consider $\frac{1}{x}$.
- there must be some $N \in \mathbb{N}$ such that $N > \frac{1}{x}$
- thus $\frac{1}{N} < x$



- But then $x \notin A_N$, by def'n of A_N
- This is a contradiction since we have $x \in A_n$ for every n.

- hence it must be that

$$\bigcap_{n \in \mathbb{N}} A_n \subseteq \{0\}$$

- Thus

$$\{0\} = \bigcap_{n \in \mathbb{N}} A_n$$

Disproving claims with counterexamples

- To show that a prop'n is false, we must prove the logical negation of the prop'n (we'll talk about logical negation in ch. 4)

- For universal claims, i.e. claims that begin "For any sets..." or "For all sets..." it suffices to produce a single counterexample.

E.g. Show the following claim is false:

Prop'n For any sets A, B, C we have that if $A \subseteq B \cup C$ then either $A \subseteq B$ or $A \subseteq C$.

- Consider the sets

$A = \{2, 3\}$
$B = \{1, 2\}$
$C = \{3, 4\}$

then $B \cup C = \{1,2,3,4\}$

so that

$$A \subseteq B \cup C$$

but

$A \not\subseteq B$, since $3 \in A$ $3 \notin B$

and $A \not\subseteq C$, since $2 \in A$ $2 \notin C$

Hence the prop'n is <u>false</u>.