Binary Relations

- Binary relations are ubiquitous in math,
- e.g. we have order relations like
  $x \leq y$ ("x is less than or equal to y")
  $x < y$ ("x is strictly less than y")
  the subset relation
  $X \subseteq Y$ ("X is a subset of Y")
  the divisibility relation
  $n \mid m$ ("n divides m")

etc...

- But what are $\leq$, $<$, $\subseteq$, $\mid$ as mathematical objects?
- It turns out to be useful to define binary relations as sets of ordered pairs.

Def'n - Suppose A, B are sets. A binary relation on A and B is a subset $R \subseteq A \times B$.
- if the pair $(a,b) \in R$ we say "a is related to b" and sometimes instead write aRb.

- A is called the <u>domain</u> of the relation R; B is called the <u>codomain</u>.
  ↳ it is often the case that $A = B$, i.e. $R \subseteq A^2$. Then we say simply that R is a <u>relation on A</u>.

<u>Examples</u> - ① Let A = set of Shakespeare's characters
B = set of Shakespeare's plays.
- Define a relation $R \subseteq A \times B$ by saying $(a, b) \in R$ iff a appears in b.
- Then (Romeo, "Romeo and Juliet") $\in R$
and (Iago, "Othello") $\in R$
but (Romeo, "Othello") $\notin R$.
- we might also write
Romeo R "Romeo and Juliet"
Iago R "Othello"
Romeo $\not R$ "Othello".

2 - Consider the relations $\leq$, $<$, and $|$ on $\mathbb{N}$.

- we can think of them as sets of ordered pairs:

$$\leq \; = \{(1,1), (1,2), (1,3), (2,3), (2,4), \dots\}$$
$$< \; = \{(1,2), (1,3), (2,3), \dots\}$$
$$| \; = \{(1,1), (2,4), (5,15), \dots\}$$

- we usually express, e.g. $(1,2) \in <$ by writing $1 < 2$. and $(2,1) \notin <$ by writing $2 \not< 1$.

- if we think of $\mathbb{N}^2$ as a grid then $\leq$ is the "lower triangular half-grid"

$$\mathbb{N}\times\mathbb{N}: \quad \begin{array}{ccccc} (4,1) & \vdots & \vdots & & \\ (3,1) & (3,2) & (3,3) & \dots & \\ (2,1) & (2,2) & (2,3) & (2,4) & \dots \\ (1,1) & (1,2) & (1,3) & (1,4) & \dots \end{array}$$

$\leq$ $\qquad$ $<$

3 Let A be a set. We can think of equality $=$ as a relation on A
$=$ is the set $\{(x,x) : x \in A\}$

Properties of Relations

Def'n Suppose A is a set and R is a relation on A (i.e. $R \subseteq A \times A$)

1. R is reflexive iff $(\forall x \in A)\,((x,x) \in R)$
2. R is symmetric iff $(\forall x, y \in A)\,((x,y) \in R \Rightarrow (y,x) \in R)$
3. R is transitive iff $(\forall x, y, z \in A)\,((x,y) \in R \wedge (y,z) \in R \Rightarrow (x,z) \in R)$
4. R is anti-symmetric iff $(\forall x, y \in A)\,((x,y) \in R \wedge (y,x) \in R \Rightarrow x = y)$

Ex's ① On any set A, the equality relation $=$ is always reflexive, symmetric and transitive

② $\leq$ (e.g., on $\mathbb{N}$) is refl., and transitive:

$(\forall n \in \mathbb{N})\,(n \leq n)$

$(\forall l, m, n \in \mathbb{N})\,(\text{if } l \leq m \text{ and } m \leq n \text{ then } l \leq n)$

- but $\leq$ is not symmetric for example $3 \leq 5$ but $5 \not\leq 3$.
- $\leq$ is antisymmetric since if $n \leq m$ and $m \leq n$ then $n = m$.

(3) $<$ (e.g. on $\mathbb{N}$) is not reflexive, symmetric, but is transitive.

(4) Let $A = \{$rock, paper, scissors$\}$
Define a relation R on A by:
$(a,b) \in R$ iff a beats b.
Then R is not transitive
(scissors, paper) $\in R$
(paper, rock) $\in R$
(scissors, rock) $\notin R$

(5) Let's consider the divisibility relation on $\mathbb{N}$:
$n \mid m$ iff n divides m
i.e. iff $\exists k \in \mathbb{N}$ $m = nk$.

Claim the divisibility relation
(i) is refl.
(ii) is not sym.
(iii) is trans
(iv) is antisymmetric

Pf. (i) For any $n \in \mathbb{N}$ we have $n \mid n$. ✓
(ii) $2 \mid 4$ but $4 \nmid 2$. ✓

(iii) Suppose $l, m, n \in \mathbb{N}$ and $l | m$ and $m | n$, i.e.

$m = k_1 l$ for some $k_1 \in \mathbb{N}$

$n = k_2 m$ for some $k_2 \in \mathbb{N}$

then $n = k_2(k_1 l) = (k_2 k_1) l$

hence $l | n$ ✓

(iv) Suppose $n, m \in \mathbb{N}$ and $n | m$ and $m | n$

then $n = k_1 m$

$m = k_2 n$

hence $n = k_1 k_2 n$

hence $k_1 k_2 = 1$

hence $k_1 = k_2 = 1$

hence $n = m$ ✓

(c) Now consider $|$ on $\mathbb{Z}$.
One can check it is still refl., trans., but it is no longer antisymmetric

e.g. $2 | -2$ and $-2 | 2$ but $-2 \neq 2$.

Equivalence Relations

Relations satisfying properties ①,②,③ have a special name:

Def'n A relation R on a set A is called an <u>equivalence relation</u> iff R is reflexive, symmetric, and transitive.

Ex's ① Let A be a set and consider the equality relation $=$. This is an equivalence relation since

$\forall x, y, z \in A$

$$x = x \quad \checkmark$$
$$x = y \Rightarrow y = x \quad \checkmark$$
$$x = y \wedge y = z \Rightarrow x = z \quad \checkmark$$

② **Recall**: the floor of a real number x, denoted $\lfloor x \rfloor$, is the unique integer n s.t. $n \leq x < n+1$

e.g.
$$\lfloor 1.3 \rfloor = 1$$
$$\lfloor -2.6 \rfloor = -3$$
$$\lfloor 5 \rfloor = 5$$

Define a relation R on $\mathbb{R}$ by
$(x,y) \in R$ iff $\lfloor x \rfloor = \lfloor y \rfloor$

<u>Claim</u>: R is an equiv. relation.
Pf (i) $\forall x \in \mathbb{R}$ we have $\lfloor x \rfloor = \lfloor x \rfloor$
hence $(x,x) \in R$.
(ii) $\forall x, y \in \mathbb{R}$, if $\lfloor x \rfloor = \lfloor y \rfloor$
then $\lfloor y \rfloor = \lfloor x \rfloor$. Hence if $(x,y) \in R$
then $(y,x) \in R$. ✓
(iii) $\forall x,y,z \in \mathbb{R}$ if $\lfloor x \rfloor = \lfloor y \rfloor$
and $\lfloor y \rfloor = \lfloor z \rfloor$ then $\lfloor x \rfloor = \lfloor z \rfloor$
Hence if $(x,y) \in R$ and $(y,z) \in R$
then $(x,z) \in R$ ✓.

(3) More generally, let $f: \mathbb{R} \to \mathbb{R}$
be any fixed function on $\mathbb{R}$.
Define a relation R_f on $\mathbb{R}$
by $(x,y) \in R_f$ iff $f(x) = f(y)$
Then R_f is an equiv. relation

<u>Pf</u>: Homework.

(4) Define a relation $\equiv_3$ on $\mathbb{Z}$ as follows.

$(n,m) \in \equiv_3$ iff $3 \mid (m-n)$

↳ we'll write $n \equiv_3 m$ for $(n,m) \in \equiv_3$.

- e.g. $2 \equiv_3 5$ since $3 \mid (5-2)$

$7 \equiv_3 -2$ since $3 \mid (-2-7)$

$6 \not\equiv_3 7$ since $3 \nmid (7-6)$

<u>Claim</u>: $\equiv_3$ is an equivalence relation on $\mathbb{Z}$.

Pf (i) For any $n \in \mathbb{Z}$ we have $3 \mid (n-n)$, i.e. $3 \mid 0$.

Hence $n \equiv_3 n$

(ii) For any $n, m \in \mathbb{Z}$ if $n \equiv_3 m$ then $3 \mid (m-n)$, i.e. $m-n = 3k$ for some $k \in \mathbb{Z}$

hence $n-m = 3(-k)$

hence $3 \mid (n-m)$, i.e. $m \equiv_3 n$ ✓

(iii) For any $n, m, l \in \mathbb{Z}$, if $n \equiv_3 m$ and $m \equiv_3 l$ then $3 \mid m-n$ and $3 \mid l-m$, i.e. $m-n = 3k_1$ and $l-m = 3k_2$ for some $k_1, k_2 \in \mathbb{Z}$

then $l - n = (l - m) + (m - n)$
$= 3k_2 + 3k_1$
$= 3(k_2 + k_1)$

hence $3 \mid l - n$

i.e. $n \equiv_3 l$.

↳ $\equiv_3$ is called <u>equivalence</u> modulo 3 and it is more conventional to write $n \equiv_3 m$ as $n \equiv m \pmod 3$.

↳ observe that $n \equiv m \pmod 3$ iff n, m yield <u>the</u> <u>same</u> <u>remainder</u> when divided by 3.

e.g. $2 \equiv 5 \pmod 3$ since $2 = 0 \cdot 3 + 2$
$5 = 1 \cdot 3 + 2$

$7 \equiv 13 \pmod 3$ since $7 = 2 \cdot 3 + 1$
$13 = 4 \cdot 3 + 1$

$7 \equiv -2 \pmod 3$ since $7 = 2 \cdot 3 + 1$
$-2 = -1 \cdot 3 + 1$

$7 \not\equiv 11 \pmod 3$ $7 = 2 \cdot 3 + 1$
$11 = 3 \cdot 3 + 2$

(5) There is nothing special about 2.
For any fixed $k \in \mathbb{N}$ can define
$\equiv_k$ on $\mathbb{Z}$ by

$n \equiv_k m$ iff $k \mid (m-n)$
(iff m, n have same remainder when divided by k)

↳ again, the standard notation is
$n \equiv m \pmod{k}$
↳ all of these "congruence modulo k" relations are equivalence relations.

<u>non examples</u>

(1) Let $\leq$ be the usual "less than or equal to" relation on $\mathbb{R}$
- then $\leq$ is <u>not</u> an equiv relation
- $\leq$ is refl. ($x \leq x$ always)
and trans. ($x \leq y \wedge y \leq z \Rightarrow x \leq z$ always)
but <u>not</u> symmetric (e.g. $3 \leq 5$ but $5 \not\leq 3$)

(2) Let $\neq$ denote the inequality relation on $\mathbb{Z}$.
- then $\neq$ is <u>not</u> an equiv relation

- $\neq$ is symmetric ($m \neq n \Rightarrow n \neq m$ always)
but not reflexive (never have $n \neq n$)
not transitive ($2 \neq 5$ and $5 \neq 2$ but $2 = 2$)

3) Define a relation R on $\mathbb{R}$ by $(x,y) \in R$ iff $|x-y| < 1$.
- then R is reflexive ($|x-x| < 1$ always)

is symmetric (if $|x-y| < 1$ then $|y-x| < 1$)

but is not transitive
e.g. $(0, 3/4) \in R$ since $|0 - 3/4| = 3/4 < 1$
$(3/4, 6/4) \in R$ since $|3/4 - 6/4| = 3/4 < 1$
but $(0, 6/4) \notin R$ since $|0 - 6/4| = 6/4 \geq 1$.

Equivalence classes.

- Suppose R is an equiv relation on a set A
- for a fixed $x \in A$, the equivalence class of x, denoted $[x]_R$ is the set of elements related to x by R:

$$[x]_R = \{y \in A \mid (x,y) \in R\}$$

Warning: overloaded notation

- we have used [ ]'s when writing $[n] = \{1, 2, \ldots, n\}$

- this is a completely different meaning than $[x]_R$ for an equiv relation R, be careful not to get confused.

Ex's ① Let $=$ be the equality relation on $\mathbb{R}$. Then for any fixed $x \in \mathbb{R}$ we have

$$[x]_= \text{ is } \{y \in \mathbb{R} \mid x = y\}$$

$$\text{i.e. } \{x\}$$

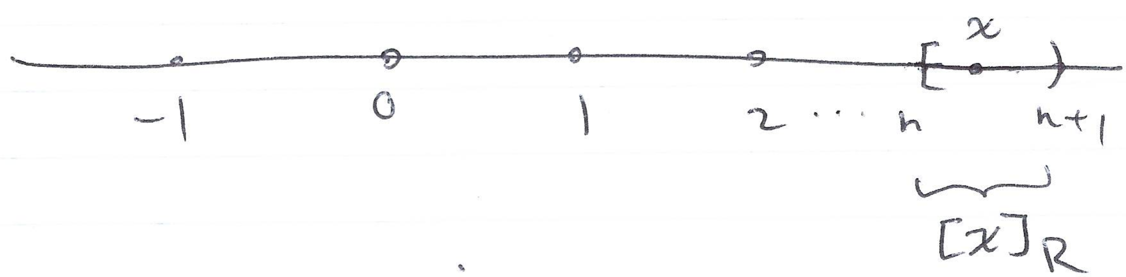
② Let R denote the floor equiv relation on $\mathbb{R}$: $(x,y) \in R$ iff $\lfloor x \rfloor = \lfloor y \rfloor$.

Observe: for a fixed $x \in \mathbb{R}$, if $\lfloor x \rfloor = n$ then

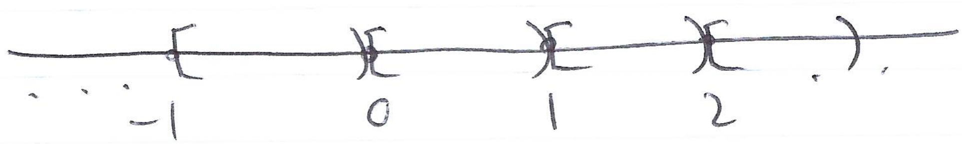
$$\begin{aligned} [x]_R &= \{y \in \mathbb{R} \mid (x,y) \in R\} \\ &= \{y \in \mathbb{R} \mid \lfloor x \rfloor = \lfloor y \rfloor\} \\ &= \{y \in \mathbb{R} \mid n = \lfloor y \rfloor\} \end{aligned}$$

$= \{y \in \mathbb{R} \mid n \leq y < n+1\}$
$= [n, n+1)$

<u>Picture</u>:



<u>Notice</u>: the equivalence classes of R partition $\mathbb{R}$:



this turns out to be typical!

Reminder: equiv class

③ Let $\equiv_3$ denote equivalence modulo 3 on $\mathbb{Z}$: $m \equiv_3 n$ iff $3 \mid n-m$

Q: what are the equivalence classes of $\equiv_3$?

Let's write some down:

$[0]_{\equiv_3} = \{n \in \mathbb{Z} \mid 0 \equiv_3 n\}$
$= \{n \in \mathbb{Z} \mid 3 \mid n-0\}$

$= \{n \in \mathbb{Z} \mid 3 \mid n\}$
$= \{\ldots, -3, 0, 3, 6, \ldots\}$

$[1]_{\equiv_3} = \{n \in \mathbb{Z} \mid 1 \equiv_3 n\}$
$= \{n \in \mathbb{Z} \mid 3 \mid n-1\}$
$= \{n \in \mathbb{Z} \mid (\exists k \in \mathbb{Z})\ n = 3k+1\}$
$= \{\ldots, -5, -2, 1, 4, 7, \ldots\}$

$[2]_{\equiv_3} = \{n \in \mathbb{Z} \mid 2 \equiv_3 n\}$
$= \{n \in \mathbb{Z} \mid 3 \mid n-2\}$
$= \{n \in \mathbb{Z} \mid (\exists k \in \mathbb{Z})(n = 3k+2)\}$
$= \{\ldots, -4, -1, 2, 5, 8, \ldots\}$

$[3]_{\equiv_3} = \{n \in \mathbb{Z} \mid 3 \equiv_3 n\}$
$= \{n \in \mathbb{Z} \mid 3 \mid n-3\}$
$= \{n \in \mathbb{Z} \mid 3 \mid n\}$
$= [0]_{\equiv_3}$

~~$[4]_{\equiv_3} = \{n \in \mathbb{Z} \mid$~~

and further $[4]_{\equiv_3} = [1]_{\equiv_3}$

$[5]_{\equiv_3} = [2]_{\equiv_3}$

$[6]_{\equiv_3} = [0]_{\equiv_3} = [3]_{\equiv_3}$

We see again that the equivalence classes of $\equiv_3$ partition $\mathbb{Z}$.

$$\mathbb{Z} = \{\ldots, -6, -3, 0, 3, 6, \ldots\} \cup \{\ldots, -5, -2, 1, 4, 7, \ldots\} \cup \{\ldots, -4, -1, 2, 5, 8, \ldots\}$$

$$= [0]_{\equiv_3} \cup [1]_{\equiv_3} \cup [2]_{\equiv_3}$$

Notation: for equivalence modulo n $\equiv_n$ we usually write $[x]_n$ instead of $[x]_{\equiv_n}$

so e.g.:

$$\mathbb{Z} = [0]_3 \cup [1]_3 \cup [2]_3 .$$

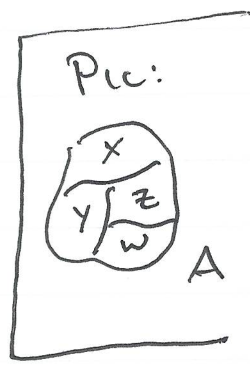
$\hookrightarrow$ we'll see how in fact "partition" and "equivalence relation" are in some sense the same idea

Recall: **Def'n** Let A be a set. A **partition** $\mathbb{P}$ of A is a collection of subsets of A (i.e. $\mathbb{P} \subseteq \mathcal{P}(A)$) such that

1. $(\forall X \in \mathbb{P})\ X \neq \emptyset$
2. $(\forall X, Y \in \mathbb{P})\ \underbrace{X \neq Y \Rightarrow X \cap Y = \emptyset}_{\text{"} X = Y \text{ or } X \cap Y = \emptyset \text{"}}$
3. $\bigcup_{X \in \mathbb{P}} X = A$

(Before, we had only defined indexed partitions, i.e. $\mathbb{P} = \{X_i : i \in I\}$ s.t.

1. $(\forall i \in I)(X_i \subseteq A \ \wedge X_i \neq 0)$
2. $(\forall i, j \in I)\ i \neq j \Rightarrow X_i \cap X_j = \emptyset$)
3. $\bigcup_{i \in I} X_i = A$)



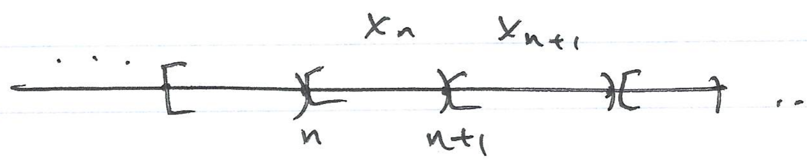
Examples ① Let

$A = \{\ldots, -3, 0, 3, \ldots\}$
$B = \{\ldots, -2, 1, 4, \ldots\}$
$C = \{\ldots, -1, 2, 5, \ldots\}$

Then $\mathbb{P} = \{A, B, C\}$ is a partition of $\mathbb{Z}$:

$A, B, C \neq \emptyset$ ✓
$A \cap B = A \cap C = B \cap C = \emptyset$ ✓
$\bigcup_{X \in \mathbb{P}} X = A \cup B \cup C = \mathbb{Z}$ ✓

② For $n \in \mathbb{Z}$, define $X_n = [n, n+1)$
Then $\mathbb{P} = \{X_n : n \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$.

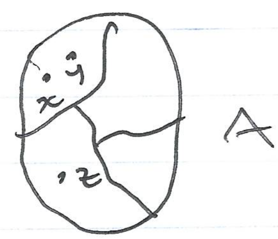


③ Let $A = \{1, 4\}$
$B = \{2, 3\}$
Then $\mathbb{P} = \{A, B\} = \{\{1,4\}, \{2,3\}\}$ is a partition of $\{1, 2, 3, 4\}$

Partitions yield equivalence relations

idea: If $\mathbb{P}$ is a partition on A can define an equiv relation R on A by rule "two el'ts are related iff they are in same piece of partition"

Picture:



$(x,y) \in R$ but $(x,z) \notin R$.

Let's prove this works.

__Theorem__: Suppose $\mathbb{P}$ is a partition of A. Define a relation $R_{\mathbb{P}}$ on A by:

$$(\forall x, y \in A) \quad (x,y) \in R_{\mathbb{P}} \text{ iff } \exists X \in \mathbb{P} \text{ s.t. } x \in X \text{ and } y \in X$$

__Then__: $R_{\mathbb{P}}$ is an equiv relation on A.

__Pf__: (i) __reflexivity__: - Fix $x \in A$. Since $\mathbb{P}$ is a partition of A, there is $X \in \mathbb{P}$ s.t. $x \in X$.

- Hence $x \in X$ as well
- hence $(x,x) \in R_{\mathbb{P}}$

(ii) __symmetry__: Fix $x, y \in A$ and suppose $(x,y) \in R_{\mathbb{P}}$. Then $\exists X \in \mathbb{P}$ with $x \in X$ and $y \in X$.

- Hence $y \in X$ and $x \in X$
- Hence $(y,x) \in R_{\mathbb{P}}$

(iii) __transitivity__: Fix $x, y, z \in A$ and suppose $(x,y) \in R_{\mathbb{P}}$ and $(y,z) \in R_{\mathbb{P}}$.

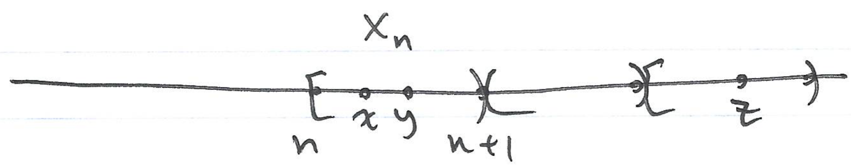
- then $\exists X \in \mathbb{P}$ s.t. $x, y \in X$
  and $\exists Y \in \mathbb{P}$ s.t. $y, z \in Y$
- hence $y \in X \cap Y$, so $X \cap Y \neq \emptyset$
- hence $X = Y$
- hence x and z are in X
- hence $(x, z) \in R_{\mathbb{P}}$. ✓

<u>ex's</u> ① Let $\mathbb{P} = \{X_n : n \in \mathbb{Z}\}$ be our partition of $\mathbb{R}$

- i.e. $X_n = [n, n+1)$
- Let $R_{\mathbb{P}}$ be the associated equiv relation: $(x, y) \in R_{\mathbb{P}}$ iff $\exists n$ s.t. $x \in X_n$ and $y \in X_n$
  i.e. $x \in [n, n+1)$ and $y \in [n, n+1)$.

↳ this is the same equiv. relation from last class, R.

- then we defined it as $(x, y) \in R$ iff $\lfloor x \rfloor = \lfloor y \rfloor$.



$(x, y) \in R_{\mathbb{P}}$
$(x, z) \notin R_{\mathbb{P}}$.

② - Let $\mathbb{P} = \{\{1,4\}, \{2,3\}\}$ be our partition of $\{1,2,3,4\}$ and let $R_{\mathbb{P}}$ be the assoc. equiv. relation

e.g. $(1,4) \in R_{\mathbb{P}}$ and $(3,2) \in R_{\mathbb{P}}$ but $(1,3) \notin R_{\mathbb{P}}$.

- Here we can explicitly write out $R_{\mathbb{P}}$ as a set:

- $R_{\mathbb{P}} = \{(1,1), (2,2), (3,3), (4,4), (1,4), (4,1), (2,3), (3,2)\}$.

↳ no rhyme or reason to this equivalence relation, but still a perfectly good one.

<u>Equiv. relations yield partitions</u>

- above we began w/ a partition $\mathbb{P}$ of A and turned it into one equiv. relation $R_{\mathbb{P}}$ by saying the <u>equiv. classes</u> of $R_{\mathbb{P}}$ were exactly the pieces X of the partition $\mathbb{P}$.

- Conversely: given an equiv relation R on A, the <u>equiv classes</u> of R always form a partition of A

<u>Def'n</u>: Let R be an equiv relation on a set A. We denote the set of equiv classes of R as A/R:

$$A/R = \{ [x]_R : x \in A \}$$

↪ "A mod R"

<u>Ex's</u> ① Consider $\equiv_3$ on $\mathbb{Z}$.

Then:

$$\mathbb{Z}/\equiv_3 = \{ \ldots, [-1]_3, [0]_3, [1]_3, [2]_3, [3]_3, \ldots \}$$

Above we checked:

$$\cdots = [-3]_3 = [0]_3 = [3]_3 = [6]_3 = \cdots$$
$$\cdots = [1]_3 = [4]_3 = \cdots$$
$$\cdots = [2]_3 = [5]_3 = \cdots$$

So really: $\mathbb{Z}/\equiv_3 = \{ [0]_3, [1]_3, [2]_3 \}$

"set of remainders" – could as well write

$$\mathbb{Z}/\equiv_3 = \{ [3]_3, [4]_3, [5]_3 \}$$

Notation: $\mathbb{Z}/\equiv_n$ is more commonly written

$\mathbb{Z}/n\mathbb{Z}$ "$\mathbb{Z}$ mod $n\mathbb{Z}$"

in general:

$$\mathbb{Z}/n\mathbb{Z} = \{ [0]_n, [1]_n, \ldots, [n-1]_n \}$$

② let R be the "floor" relation on $\mathbb{R}$: $(x,y) \in R$ iff $\lfloor x \rfloor = \lfloor y \rfloor$.

- $\mathbb{R}/R = \{ [x]_R : x \in \mathbb{R} \}$
- we knew from before the equiv classes are intervals of the form $[n, n+1)$.
- if $x \in [n, n+1)$ then $[x]_R = [n, n+1)$ and any number in this interval serves equally well as a representative:
- so can write:

$$\mathbb{R}/R = \{ \ldots, [-1]_R, [0]_R, [1]_R, \ldots \}$$
$$= \{ \ldots, [-1,0), [0,1), [1,2), \ldots \}$$

or $= \{ \ldots, [-1/2]_R, [1/2]_R, [3/2]_R, \ldots \}$

↳ in both examples the set of equiv classes forms a partition.
↳ this is always the case.

Theorem If R is an equiv relation on a set A, then A/R is a partition of A

Pf: HW. For a hint, see problem 6.7.13 pg. 449 which outlines the proof.

Order relations

- another important type of relation is an order relation.
- come in several flavors: nonstrict/strict and partial/total.
- first need a new property!

memorize

Def'n a relation R on a set A is called irreflexive iff $(\forall x \in A)\ (x,x) \notin R$.

- e.g. $<$ and $\neq$ are irreflexive on $\mathbb{R}$, since we never have $x < x$ or $x \neq x$.

Def'n - A relation R on a set A is called a partial order on A iff R is reflexive, transitive, and antisymmetric.

- if R is a partial order on A we say that the pair (A, R) is a partially ordered set or poset.

ex's: ① $\leq$ is a partial order on $\mathbb{R}$: $\forall x, y, z \in \mathbb{R}$ we have

$x \leq x$ ✓

$x \leq y \wedge y \leq z \Rightarrow x \leq z$ ✓

$x \leq y \wedge y \leq x \Rightarrow x = y$ ✓

So $(\mathbb{R}, \leq)$ is a poset.

② Let A be any set. Then the subset relation $\subseteq$ on $P(A)$ is a partial order:

$\forall X, Y, Z \in P(A)$ we have

① $X \subseteq X$ ✓

② $X \subseteq Y \wedge Y \subseteq Z$ then $X \subseteq Z$ ✓

③ $X \subseteq Y \wedge Y \subseteq X$ then $X = Y$ ✓

So $(P(A), \subseteq)$ is a poset.

③ - We showed the divisibility relation $|$ on $\mathbb{N}$ is refl., trans., antisymmetric. Hence $(\mathbb{N}, |)$ is a poset.
- also showed $|$ is not antisymmetric on $\mathbb{Z}$. Hence $(\mathbb{Z}, |)$ is not a poset.

↳ these three examples seem to be of very different kinds

↳ and yet: any theorems that depend only on properties of refl., trans., and antisymmetry must hold for all three! (and any other poset.)

↳ all of these are nonstrict partial orders

New defn: ✱ irreflexive
e.g.

Def'n: a relation R on a set A is called a strict partial order if R is irreflexive, transitive, and antisymmetric.

↳ Before examples, an observation.
How can R be irreflexive and antisymmetric

$\forall x \in A \ (x,x) \notin R$

$\forall x,y \in A \ [(x,y) \in R \wedge (y,x) \in R \Rightarrow x = y]$

- only if $(x,y) \in R \wedge (y,x) \in R$ is never true.
- i.e. if $(x,y) \in R \Rightarrow (y,x) \notin R$.

ex's ① $<$ is a strict partial order on $\mathbb{R}$: $\forall x,y,z \in \mathbb{R}$:

① $x \not< x$ ✓

② $x < y \wedge y < z \Rightarrow x < z$ ✓

③ $\underbrace{x < y \wedge y < x}_{\text{always false}} \Rightarrow x = y$ ✓

could just observe
$x < y \Rightarrow y \not< x$.

② $\subsetneq$ is a strict partial order on $P(A)$ for any set A: $\forall X,Y,Z \in P(A)$ we have:

(i) $\neg(X \subsetneq X)$ ✓

(ii) $X \subsetneq Y \wedge Y \subsetneq Z \Rightarrow X \subsetneq Z$

let's actually prove (ii)

- since $X \subsetneq Y$ we have $X \subseteq Y$ and $\exists y \in Y$ s.t. $y \notin X$.
- since $Y \subsetneq Z$ we have $Y \subseteq Z$ and $\exists z \in Z$ s.t. $z \notin Y$
- hence $X \subseteq Z$ and also $z \notin X$.
- hence $X \subsetneq Z$ ✓

(iii) if $X \subsetneq Y$ then $\neg(Y \subsetneq X)$

pf: $\exists y \in Y$ s.t. $y \notin X$. ✓

Nonexamples: ① $\leq$ is not a strict p.o. on $\mathbb{R}$; $\subseteq$ is not a strict p.o. on $P(A)$: irreflexivity fails

② $<$, $\subsetneq$ are not partial orders. reflexivity fails.

③ $\neq$ is neither a partial or strict partial order on $\mathbb{R}$: transitivity fails: $2 \neq 5 \wedge 5 \neq 2$ but $2 = 2$. (antisymmetry, reflexivity also fails)

Total orders

Def'n: a relation R on a set A is said to be total if $(\forall x,y\in A)\,(xRy \vee yRx \vee x=y)$

Def'n (i) if R is a partial order on A that is also total then R is called a total order on A.

(ii) if R is a strict partial order on A that is also total, then R is called a strict total order on A.

Ex's ① $\leq$ is a total order on $\mathbb{R}$: we know already $\leq$ is a p.o. and we know: $(\forall x,y\in\mathbb{R})\,(x\leq y \vee y\leq x \vee x=y)$

② $\subseteq$ is not a total order on : $P(\mathbb{N})$.

e.g. if $X=\{1,2,3\}$ $Y=\{3,4\}$ then $X\nsubseteq Y$ and $Y\nsubseteq X$ and $X\neq Y$.

③ $<$ is a strict total order on $\mathbb{R}$: $(\forall x,y)\,(x<y \vee y<x \vee x=y)$

Two strict orders on $\mathbb{N}\times\mathbb{N}$

① Define a relation R on $\mathbb{N}\times\mathbb{N}$ by $(n_1,m_1)\ R\ (n_2,m_2)$ iff $n_1 < n_1$ and $m_1 < m_2$.

e.g. $(1,2)\ R\ (3,5)$ since $1<3$ and $2<5$

but $(3,1) \not R (2,2)$ since $3 \not< 2$.

Claim: R is a strict partial order

Pf: Fix (n_1,m_1), (n_2,m_2), (n_3,m_3) in $\mathbb{N}\times\mathbb{N}$

- then $(n_1,m_1) \not R (n_1,m_1)$ so irreflexivity holds (why: $n_1 \not< n_1$)
- if $(n_1,m_1)\ R\ (n_2,m_2)$ and $(n_2,m_2)\ R\ (n_3,m_3)$

then $n_1 < n_2$ and $m_1 < m_2$

and $n_2 < n_3$ and $m_2 < m_3$

hence by transitivity of $<$

$n_1 < n_3$ and $m_1 < m_3$

hence $(n_1,m_1)\ R\ (n_3,m_3)$

so transitivity holds ✓

- if $(n_1, m_1) \, R \, (n_2, m_2)$ then $n_1 < n_2$ so $(n_2, m_2) \not R \, (n_1, m_1)$ so <u>antisymmetry</u> holds ✓

<u>Observe</u>: R is not total: e.g. $(1,3) \not R \, (2,1)$, $(2,1) \not R \, (1,3)$ and $(1,3) \neq (2,1)$.

② (The lexicographic order on $\mathbb{N} \times \mathbb{N}$)
Define a relation R on $\mathbb{N} \times \mathbb{N}$ by the rule $(n_1, m_1) \, R \, (n_2, m_2)$ iff $n_1 < n_2$ <u>or</u> $(n_1 = n_2$ and $m_1 < m_2)$

e.g.
- $(5,2) \, R \, (7,1)$ because $5 < 7$
- $(5,2) \, R \, (7,12)$
  $5 < 7$
- $(5,2) \, R \, (5,3)$
  $5 = 5 \quad 2 < 3$
- $(5,1) \not R \, (4,3)$
  $5 > 4$
- $(5,7) \not R \, (5,3)$
  $5 = 5$ but $7 > 3$

"dictionary ordering"

<u>Claim</u> R is a strict total order on $\mathbb{N}\times\mathbb{N}$

<u>Pf</u>: Fix (n_1,m_1), (n_2,m_2), (n_3,m_3) in $\mathbb{N}\times\mathbb{N}$.

(i) Then $(n_1,m_1) \not R (n_1,m_1)$ because $n_1=n_1$ and $m_1=m_1$. So irreflexivity holds

(ii) Suppose $(n_1,m_1) R (n_2,m_2)$
$(n_2,m_2) R (n_3,m_3)$

<u>Case 1</u>: $\boxed{n_1<n_2}$. – Observe that since $(n_2,m_2) R (n_3,m_3)$ we have $n_2 \leq n_3$.

– hence $n_1<n_3$ (by trans. of $\leq$)

– hence $(n_1,m_1) R (n_3,m_3)$ in this case

<u>Case 2</u>: $\boxed{n_1=n_2}$ Then $m_1<m_2$

<u>Subcase 1</u>: $\underline{n_2<n_3}$

– Then since $n_1=n_2$ we have $n_1<n_3$.

– hence $(n_1,m_1) R (n_3,m_3)$

<u>Subcase 2</u>: $\underline{n_2=n_3}$

– Then it must be $m_2<m_3$

– hence $m_1<m_3$

– hence $(n_1,m_1) R (n_3,m_3)$

what does $\mathbb{N}\times\mathbb{N}$ look like under this order?

$\bullet \; \bullet \; \bullet \; \cdots \; \bullet \; \bullet \; \bullet \; \cdots$

hence in all cases

$(n_1,m_1)\, R\, (n_3,m_3)$

and transitivity holds.

(iii) – Suppose $(n_1,m_1) R (n_2,m_2)$.

Case 1: $\boxed{n_1<n_2}$ Then $(n_2,m_2) \not R (n_1,m_1)$

Case 2: $\boxed{n_1=n_2}$ and $\boxed{m_1<m_2}$. Then $(n_2,m_2) \not R (n_1,m_1)$ in this case as well.

– hence in all cases $(n_2,m_2) \not R (n_1,m_1)$ and antisymmetry holds

(iv) Suppose $(n_1,m_1) \neq (n_2,m_2)$
and $(n_1,m_1) \not R (n_2,m_2)$

then cannot be that $n_1<n_2$

hence $n_2 \leq n_1$

– if $n_2<n_1$ then $(n_2,m_2) R (n_1,m_1)$

– if $n_2=n_1$ then cannot be that $m_1<m_2$ or $m_1=m_2$.

– hence $m_2<m_1$ and $(n_2,m_2) R (n_1,m_1)$

– hence in all cases $(n_2,m_2) R (n_1,m_1)$ and totality holds ✓