Ch. 4 : Intro to Mathematical Logic

Goals: - Learn how to write more formal statements and proofs

- expand our repertoire of proof techniques

↳ much of this will consist of introducing symbols for words/phrases used in making statements

Recall: Def'n (intuitive): A mathematical statement (or prop'n) is a grammatically correct declarative sentence, consisting of words and symbols, that is either true or false

↳ to rigorously define "statement" requires more background in formal logic where statements are entirely symbolic.

↳ "grammatically correct" has a definite meaning in that context.

Ex's

① Every integer is a real number (T)

② Every real number is an integer (F)

③ There exists $x \in \mathbb{R}$ s.t. $x \notin \mathbb{Z}$ (T)

④ $1+2=3$ (T)

⑤ Every integer greater than 5 can be written as the sum of three primes (unknown; but T/F)

Nonex's

① $\notin 3\pi$ (not grammatically correct / meaningless)

② Shakespeare (not a declarative sentence / no truth value)

③ $x^2+1=2$

↳ this is a meaningful sequence of symbols, but no truth value unless x is specified (or quantified over — more on this later).

↳ this is an example of a variable proposition: a sentence that becomes a statement once its variables are specified.

↳ will denote statements P, Q, S, .. etc. and var prop'ns $P(x)$, $Q(y,z)$, .. etc.

↳ e.g. might say:

- Let P be the statement "$5^2+1=2$" (F)
- Let $Q(x)$ be the var prop'n "$x^2+1=2$"

↳ then $Q(5)$ is the statement "$5^2+1=2$" (F)

↳ $Q(1)$ is the statement "$1^2+1=2$" (T)

Ex's of var. prop'ns

1. $x^2+1 \leq 0$
2. $x \in \mathbb{R}$ and $x<3$
3. $z = x^2+y$

↳ this has multiple variables

↳ always indicate all vars when denoting var prop'ns e.g. could say: let $Q(x,y,z)$ be "$z = x^2 + y$"

- Then $Q(2,1,1)$ is T, $Q(2,3,4)$ is F.

$\boxed{\text{Quantifying Variables}}$

↳ other way to turn a var prop'n into a statement is to quantify over its variables

↳ e.g. "$x^2+1=2$" is a var. prop'n but

"There exists $x \in \mathbb{R}$ such that $x^2+1=2$" is a statement. (T)

as is

"For every $x \in \mathbb{R}$, $x^2+1=2$" (F)

- The clauses "For every $x \in S$ ..." and "There exists $x \in S$ such that ..." are examples of quantification of the variable x.

- We introduce the symbols:

$\forall$ stands for "for all"

$\exists$ stands for "there exists"

↳ $\forall$ is called the <u>universal</u> <u>quantifier</u>
$\exists$ is called the <u>existential</u> <u>quantifier</u>.

- Given a var. prop'n $P(x)$ and a set S, the sentences
  "For every $x \in S$ we have $P(x)$"
  "There exists $x \in S$ such that $P(x)$"
  are <u>statements</u>.

- write these symbolically as

$$\forall x \in S.\ P(x) \quad \text{or} \quad (\forall x \in S)\ P(x)$$
$$\exists x \in S.\ P(x) \quad\quad (\exists x \in S)\ P(x)$$

parentheses clearly separate clauses, but too many parentheses = clutter

<u>Ex's</u>

① $\underbrace{(\exists x \in \mathbb{N})}_{\text{quantifier}}\ \underbrace{(x > 5)}_{P(x)}$

"There is a natural number greater than 5." (T)

② $(\forall x \in \mathbb{N}).\ (x > 5)$

"Every natural number is greater than 5." (F)

③ $(\forall x \in \mathbb{N})(x > 0)$ (T)

④ $(\forall x \in \mathbb{Z})(x > 0)$ (F)

↳ these ex's show that the set we quantify over matters.

<u>Quantifying over multiple variables</u>

① $(\forall x, y \in \mathbb{N})(x + y \geq 2)$ (T)

– read this as: "for all x and y in $\mathbb{N}$, $x + y \geq 2$"

↳ we can also nest $\forall$s and $\exists$s, but in this case the order of quantifiers is <u>very</u> <u>important</u>.

② $(\forall x \in \mathbb{N})(\exists y \in \mathbb{R})(x = y^2)$

"For every $x \in \mathbb{N}$, there is $y \in \mathbb{R}$, such that $x = y^2$" i.e.
"Every natural number has a real square root." (T)

③ $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x = y^2)$

"Every real number has a real square root." (F)

(4) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{C})(x = y^2)$

(T)

↳ Question: what happens if we reverse the quantifiers in (2)?
↳ A: completely changes meaning of statement.

(5) $(\exists y \in \mathbb{R})(\forall x \in \mathbb{N})(x = y^2)$

"There is a real number y s.t. every natural number equals y^2."

↳ This is a perfectly well-written mathematical statement, but is absurd (and definitely false)

↳ moral: order of quantifiers a big deal!

(D)

(6) "Inside" quantifiers: the following is also a well-written sentence:

$$(\forall x \in \mathbb{R})(\text{if } x \geq 0, \text{ then } (\exists y \in \mathbb{R})(y^2 = x))$$

Note: Quantifying set variables

↳ we have insisted quantified variables range over a specified set

↳ e.g.

$(\forall x \in \mathbb{R})(x^2 \geq 0)$ is meaningful

$(\forall x).(x^2 \geq 0)$ is not.

↳ a problem arises when we wish to quantify over variables that stand for <u>sets</u>.

↳ e.g. if we wish to write "For every set S, $\emptyset \subseteq S$" symbolically, might try:

$$\forall S \in (\dots).\ \phi \subseteq S$$

collection of all sets??

- But the collection of all sets is not a set (Russell's Paradox)

- We get around this by writing statements like
  "For every set S, ..."
  "There is a set S ..."
  using words instead of $\forall$, $\exists$.

Connectives and Truth Tables

- Connectives are symbols that allow us to combine multiple statements into a single, longer statement.
- all connectives we'll study are binary (combine two statements into one) except negation, which is unary.
- Truth Tables tell us how the truth of the connected statement depends on the truth of the original statements.

Conjunction ("and")

- If P, Q are statements, the conjunction of P and Q is $P \wedge Q$ ("P and Q")
- $P \wedge Q$ is true iff both P and Q are true

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Ex's ① - Let P be

$$(\forall x \in \mathbb{R})(x+1 > x)$$

- Let Q be

2 is a prime number

- Let R be

4 is a prime number

Then:

$P \wedge Q$ is true (both P, Q true)

but

$P \wedge R$, $Q \wedge R$ both false (since R is false)

② Written out, $P \wedge Q$ is

$((\forall x \in \mathbb{R})(x>1)) \wedge$ (2 is a prime number)

↳ technically, parentheses should not be part of our expression, but we'll use them sometimes to make reading easier.

Disjunction ("or")

- the disjunction of P, Q is written $P \vee Q$ ("P or Q")
- is true iff at least one of P, Q is true

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Ex's

① $\underbrace{(2 \text{ is prime})}_{P} \vee \underbrace{(4 \text{ is prime})}_{Q}$ is true

② (4 is prime) $\vee$ (6 is prime) is false

Negation

↳ negation is our only unary connective

↳ negation of P is $\neg P$ ("not P")

↳ is true iff P is false

P	$\neg P$
T	F
F	T

ex's ① $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y^2 = x)$

is false

② $\neg(\forall x \in \mathbb{N}.\ \exists y \in \mathbb{N}.\ y^2 = x)$

is true

③ For any statement P, the statement $P \vee \neg P$ is true

↳ e.g.

(6 is prime) $\vee\ \neg$(6 is prime)

is true.

More examples

① Can also use connectives in var prop'ns. As before, connected var prop'ns become statements only once their variables are specified or quantified.

↳ e.g. let $P(x)$ be "$(x>0) \wedge (x \text{ is odd})$"

↳ Then:

$P(5)$ is true.

↳ whereas

$\forall x \in \mathbb{N}. P(x)$ is false (i.e. $\forall x \in \mathbb{N}. (x>0) \wedge (x \text{ is odd})$)

② $\forall x \in \mathbb{R}. \left(x \leq 0 \vee (\exists y \in \mathbb{R})(x = y^2) \right)$

is true.

③ - We can use connectives in def'ns, set-builder notation, etc.

- e.g. if A and B are subsets of a universal set U, then:

$$A \cap B = \{ x \in U \mid x \in A \wedge x \in B \}$$
$$A \cup B = \{ x \in U \mid x \in A \vee x \in B \}$$

Implication

- Given statements P, Q the statement $P \Rightarrow Q$ is read "P implies Q" or "if P, then Q"
- it is true iff whenever P is true, Q is also true

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

- most confusing connective
- notice $P \Rightarrow Q$ is automatically true if P is false
- $P \Rightarrow Q$ is only false when P is true and Q is false

↳ statements of the form $P \Rightarrow Q$ are called conditional statements

ex: (1) $\overset{T}{1+1=2} \Rightarrow \overset{T}{1+1+1=3}$

is true

(2) $\overset{T}{1+1=2} \Rightarrow \overset{T}{(\forall x \in \mathbb{R})(x^2 \geq 0)}$

is <u>true</u> (even though P, Q unrelated)

(3) $\overset{F}{\exists x \in \mathbb{R} .\, x^2 = -1} \Rightarrow \overset{F}{1+1=3}$

is <u>true</u>

so is:

$\exists x \in \mathbb{R} .\, x^2 = -1 \Rightarrow 1+1=2$

(4) $1+1=2 \Rightarrow \exists x \in \mathbb{R} .\, x^2 = -1$

is <u>false</u>

(5) can also use $\Rightarrow$ in var prop'ns

e.g.

$x \geq 2 \Rightarrow x^2 \geq 4$

is a well-written var prop'n and

$\forall x \in \mathbb{R} . (x \geq 2 \Rightarrow x^2 \geq 4)$

is a <u>true</u> statement.

(6) $\forall x \in \mathbb{R} .\, (x^2 \geq 4 \Rightarrow x \geq 2)$

is <u>false</u>.

because there is a real number x (e.g. $x = -3$) s.t. $x^2 \geq 4$ is true but $x \geq 2$ is false

Equivalence

Given P, Q, the statement $P \Leftrightarrow Q$, read "P if and only if Q" (or "P iff Q") is true iff P and Q have the same truth value

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

– Statements of the form $P \Leftrightarrow Q$ are called <u>biconditional</u> <u>statements</u>

<u>ex's</u> ① $1+1=2 \Leftrightarrow 2+2=4$ is <u>true</u>

② $1+1=3 \Leftrightarrow 2+2=5$ is <u>true</u>

③ $1+1=2 \Leftrightarrow 2+2=5$ is <u>false</u>

④ $\forall x \in \mathbb{R}.\,(x \geq 0 \Leftrightarrow \exists y \in \mathbb{R}.\, y^2 = x)$ is <u>true</u>

↳ Why: for every real x, the statements "$x \geq 0$" "$\exists y \in \mathbb{R}.\, y^2 = x$" are either <u>both</u> <u>true</u> or <u>both</u> <u>false</u>.

<u>Def'n</u> Statements P, Q are called <u>logically equivalent</u> iff $P \Leftrightarrow Q$ is true.

↳ ex's ① and ② give trivial examples of logically equiv. statements (2 true statements are always logically equiv.; 2 false statements are always logically equiv.)
↳ more interested in finding logically equivalent forms for compound statements, especially negated statements.

<u>Negation of Quantified Statements</u>

- Suppose $P(x)$ is a var. prop'n and S is a set.
- Consider the statement $\forall x \in S.\ P(x)$
- The negation is

$$\neg \forall x \in S.\ P(x)$$

- the negation is true iff it is not the case that every $x \in S$ satisfies $P(x)$

- that is, if there exists and $x \in S$ such that $\neg P(x)$

- this means
$$\neg \forall x \in S.\ P(x)$$
is logically equiv. to
$$\exists x \in S.\ \neg P(x)$$

- more succinctly, we have that no matter what $P(x)$ is,

$$\neg \forall x \in S.\ P(x) \Leftrightarrow \exists x \in S.\ \neg P(x)$$

is true

- Similarly, consider $\exists x \in S.\ P(x)$
- negation is $\neg \exists x \in S.\ P(x)$
- is true iff every $x \in S$ does not satisfy $P(x)$
- i.e. if $\forall x \in S.\ \neg P(x)$

- We have shown that no matter what $P(x)$ is

$\neg \exists x \in S.\ P(x)$ is logically equiv to
$\forall x \in S.\ \neg P(x)$

or more succinctly that

$$\neg \exists x \in S . P(x) \Leftrightarrow \forall x \in S . \neg P(x)$$

is <u>true</u>.

- Let's collect these into a theorem

<u>Thm</u> For any var prop'n $P(x)$, and set S, the following logical equivalences hold:

① $\neg \forall x \in S . P(x) \Leftrightarrow \exists x \in S . \neg P(x)$

② $\neg \exists x \in S . P(x) \Leftrightarrow \forall x \in S . \neg P(x)$

<u>Ex's</u> ① $\neg(\forall x \in \mathbb{R} . x \in \mathbb{N})$ "not all reals are naturals"

is equiv. to

$\exists x \in \mathbb{R} . \neg(x \in \mathbb{N})$ "there is a real which is not a natural."

Note: we will abbreviate

"$\neg(x \in S)$" as $x \notin S$

and "$\neg(x = y)$" as $x \neq y$

So find statement above can be written

$\exists x \in \mathbb{R} . x \notin \mathbb{N}$ (this is <u>true</u>)

② $\neg(\exists x \in \mathbb{R}.\ x \in \mathbb{N})$ "There is no real which is a natural"

$\iff \forall x \in \mathbb{R}.\ x \notin \mathbb{N}$ "Every real is not a natural"

holds

(in this case, both statements false)

③ For nested quantifiers, just iterate this process, e.g.

$$\neg(\forall x \in \mathbb{R}.\ \exists y \in \mathbb{R}.\ xy = 1)$$ "not every real has a multiplicative inverse."

$\iff \exists x \in \mathbb{R}.\ \neg(\exists y \in \mathbb{R}.\ xy = 1)$

$\iff \exists x \in \mathbb{R}.\ \forall y \in \mathbb{R}.\ xy \neq 1$ "there is an $x \in \mathbb{R}$ that has no inverse"

(these are all true since $x = 0$ has no inverse).

Negating Conjunctions, disjunctions, and negations

Theorem For any statements P, Q, the following logical equivalences hold:

(1) $\neg\neg P \Leftrightarrow P$ "not, not P" $\Leftrightarrow$ "P"

(2) $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$ "not, P and Q" $\Leftrightarrow$ "either not P, or not Q."

(3) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ "not, P or Q" $\Leftrightarrow$ "neither P, nor Q."

↳ these equivs all make intuitive sense, but to prove them we use truth tables

Pf. (1)

P	$\neg P$	$\neg\neg P$	$\neg\neg P \Leftrightarrow P$
T	F	T	T
F	T	F	T

} $\neg\neg P \Leftrightarrow P$ always true

hence $\neg\neg P \Leftrightarrow P$ always holds

② P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

…	$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
	T
	T
	T
	T

hence $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$ always holds ✓

③ Similar, try it yourself...

Note: ② and ③ are called <u>De Morgan's Laws</u> for logic.

<u>Def'n</u> a statement P is in <u>positive form</u> if any negation symbols in P apply only to substatements ~~that~~ contain no quantifiers or connectives

Ex's

① $\neg\neg(1+1=2)$
is equiv. to
$1+1=2$ (both true)

② $\neg(1+1=2 \wedge 1+1=3)$
is equiv. to
$\neg(1+1=2) \vee \neg(1+1=3)$
which we can write
$1+1 \neq 2 \vee 1+1 \neq 3$ (both true)

③ $\neg(\pi \in \mathbb{N} \vee \pi \in \mathbb{R})$
is equiv. to
$\pi \notin \mathbb{N} \wedge \pi \notin \mathbb{R}$ (both false)

Can also apply equivalences inside parentheses or quantifiers, e.g.

④ $\forall x \in \mathbb{R}. \neg(x < 0 \wedge (\exists y \in \mathbb{R}. y^2 = x))$

$\Leftrightarrow \forall x \in \mathbb{R} [\neg(x<0) \vee \neg(\exists y \in \mathbb{R}. y^2 = x)]$

$\Leftrightarrow \forall x \in \mathbb{R}. [x \geq 0 \vee (\forall y \in \mathbb{R}. y^2 \neq x)]$

Other useful equivalences

Theorem For any statements P, Q, the following equivalences hold:

① $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

② $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$

③ $(P \Leftrightarrow Q) \Leftrightarrow (P \Rightarrow Q \wedge Q \Rightarrow P))$

$\neg(P \Rightarrow Q)$	$P \wedge \neg Q$
$\neg(P \Leftrightarrow Q)$	$(P \wedge \neg Q) \vee (Q \wedge \neg P)$

Pf ① + ②

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg P \vee Q$	$\neg Q \Rightarrow \neg P$
T	T	T	F	F	T	T
T	F	F	F	T	F	F
F	T	T	T	F	T	T
F	F	T	T	T	T	T

. . .

$(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$	$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$
T	T
T	T
T	T
T	T

<u>Theorem</u> The following equivalences hold:

① $\neg(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

② $\neg(P \Leftrightarrow Q) \Leftrightarrow [(P \wedge \neg Q) \vee (Q \wedge \neg P)]$

<u>Pf</u>: Instead of using truth tables, let's employ our previous equivalences.

① $\neg(P \Rightarrow Q) \Leftrightarrow \neg(\neg P \vee Q)$

$\Leftrightarrow \neg\neg P \wedge \neg Q$ De Morgan

$\Leftrightarrow P \wedge \neg Q$ ✓

② $\neg(P \Leftrightarrow Q) \Leftrightarrow \neg[P \Rightarrow Q \wedge Q \Rightarrow P]$

$\Leftrightarrow \neg[(\neg P \vee Q) \wedge (\neg Q \vee P)]$

$\Leftrightarrow \neg(\neg P \vee Q) \vee \neg(\neg Q \vee P)$

$\Leftrightarrow (\neg\neg P \wedge \neg Q) \vee (\neg\neg Q \wedge \neg P)$

$\Leftrightarrow (P \wedge \neg Q) \vee (Q \wedge \neg P)$

Ex's Let E, O, P be the sets of even, odd, and prime positive ~~numbers~~ integers.

① Let's find the logical negation of the (false) statement

$$\forall x \in \mathbb{N}. (x \in P \Rightarrow x \in O)$$

this is:

$$\neg \forall x \in \mathbb{N} (x \in P \Rightarrow x \in O)$$

$$\Leftrightarrow \exists x \in \mathbb{N}. \neg (x \in P \Rightarrow x \in O)$$

$$\Leftrightarrow \exists x \in \mathbb{N}. \neg (\neg (x \in P) \vee x \in O)$$

$$\Leftrightarrow \exists x \in \mathbb{N}. (x \in P \wedge x \notin O)$$

(true).

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2) Let's find the logical negation of the (true) statement

$$\forall x \in \mathbb{R} . \left(x \geq 0 \Leftrightarrow \exists y \in \mathbb{R} . y^2 = x \right)$$

this is:

$$\neg \forall x \in \mathbb{R} \left(x \geq 0 \Leftrightarrow \exists y \in \mathbb{R} . y^2 = x \right)$$

$$\Leftrightarrow \exists x \in \mathbb{R} . \left[\left(\neg (x \geq 0) \wedge \exists y \in \mathbb{R} . y^2 = x \right) \vee \left(x \geq 0 \wedge \neg \exists y \in \mathbb{R} . y^2 = x \right) \right]$$

$$\Leftrightarrow \exists x \in \mathbb{R} . \left[\left((x < 0) \wedge \exists y \in \mathbb{R} . y^2 = x \right) \vee \left(x \geq 0 \wedge \forall y \in \mathbb{R} . y^2 \neq x \right) \right].$$

③ Try it yourself.

Note: - we'll use this equivs in proofs

- ① says: to prove $P \Rightarrow Q$, prove either P fails, or Q holds
- ② says: to prove $P \Rightarrow Q$, prove that if Q fails, then P fails
- ③ says: to prove $P \Leftrightarrow Q$, prove P implies Q, and Q implies P.

contrapositive

ex's Let E and O denote the sets of even and odd positive integers

① $5 \in O \Rightarrow 6 \in E$

is equiv to

$\neg(5 \in O) \vee 6 \in E$

which we can write

$5 \notin O \vee 6 \in E$

② $\forall x \in \mathbb{N}. (x \in O \Rightarrow x+1 \in E)$

is equiv to

$\forall x \in \mathbb{N}. (x \notin O \vee x+1 \in E)$

or, using ②

$\forall x \in \mathbb{N} (x+1 \notin E \Rightarrow x \notin O)$

(3) Let P denote the set of primes.
Then:

$$\forall x \in \mathbb{N}. (x \in P \Leftrightarrow x \in O)$$

is equiv. to

$$\forall x \in \mathbb{N}. [(x \in P \Rightarrow x \in O) \wedge (x \in O \Rightarrow x \in P)]$$

Associative and Distributive Laws

Theorem For any statements P, Q, R the following logical equivalences hold.

(1) $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$
(2) $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$

(3) $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$
(4) $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

– For proofs see 4.6.3 and 4.6.4 in textbook

– try to justify these to yourself intuitively.

Proving equality of sets using $\Leftrightarrow$'s

↳ there is a strong analogy between logical connectives and the set operations of Ch. 3

Connective	Set operation
$P \wedge Q$	$A \cap B$
$P \vee Q$	$A \cup B$
$P \Rightarrow Q$	$A \subseteq B$
$P \Leftrightarrow Q$	$A = B$
$\neg P$	$\overline{A}$

↳ What do we mean by this? Let's see some examples.
↳ these examples introduce a new technique: using $\Leftrightarrow$s to prove equality of two sets.

Theorem Suppose A, B are sets and U is a universal set with $A, B \subseteq U$.
Then:

(1) $\bar{\bar{A}} = A$ looks like: $\neg\neg P \Leftrightarrow P$

(2) $\overline{A \cap B} = \bar{A} \cup \bar{B}$ $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

(3) $\overline{A \cup B} = \bar{A} \cap \bar{B}$ $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

Pf: (1) Fix $x \in U$.

then:

$$
\begin{aligned}
x \in \bar{\bar{A}} &\Leftrightarrow x \notin \bar{A} && \text{def'n of complement} \\
&\Leftrightarrow \neg(x \in \bar{A}) \\
&\Leftrightarrow \neg(\neg(x \in A)) && \text{def'n of complement} \\
&\Leftrightarrow x \in A && \neg\neg P \Leftrightarrow P
\end{aligned}
$$

this chain of equivalences shows:

$x \in \bar{\bar{A}} \Leftrightarrow x \in A$

hence

$\bar{\bar{A}} = A$ ✓ $\longrightarrow$ $x \in \bar{\bar{A}} \Rightarrow x \in A$

$x \in A \Rightarrow x \in \bar{\bar{A}}$

(2) Fix $x \in U$.

then:

$$
\begin{aligned}
x \in \overline{A \cap B} &\Leftrightarrow \neg(x \in A \cap B) && \text{def'n of comp.} \\
&\Leftrightarrow \neg(x \in A \wedge x \in B) && \text{def'n of } \cap \\
&\Leftrightarrow x \notin A \vee x \notin B && \text{De Morgan} \\
&\Leftrightarrow x \in \bar{A} \vee x \in \bar{B} && \text{def'n of comp} \\
&\Leftrightarrow x \in \bar{A} \cup \bar{B}
\end{aligned}
$$

hence $x \in \overline{A \cap B} \iff x \in \bar{A} \cup \bar{B}$

thus $\overline{A \cap B} = \bar{A} \cup \bar{B}$. ✓

Pf of 3 similar . . .

Exercise: use the distributive law $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$ to prove the following theorem.

Theorem: For any sets A, B, C we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

More negation practice

↳ previous rules show how to negate $\forall$'s, $\exists$'s, $\wedge$'s, $\vee$'s, or $\neg$'s.

↳ so how do we negate $\Rightarrow$'s and $\iff$'s?

Proof Writing

Approaches: when trying to prove a statement P, can either prove P directly or assume $\neg P$ and derive a contradiction (i.e. prove $\neg\neg P$)

— More generally, can prove any statement logically equiv to P, or prove the falsity of any statement logically equiv to $\neg P$

Existence Proofs

General form: $\exists x \in S : P(x)$

Direct proof strategy: define an el't $y \in S$ and show $P(y)$ holds

Examples ① Prop'n There is an even number that can be written as the sum of two primes in two distinct ways.

PF: - Consider $n = 10$. Then n is even.
- Further $n = 5+5$ and $n = 7+3$.
- Since 3, 5, 7 are primes, the proposition is proved.

Prop'n
(2) There exist two irrational numbers x, y s.t. x^y is rational.

↳ we can write this statement symbolically: $\exists x, y \in \mathbb{R} . (x, y \notin \mathbb{Q} \wedge x^y \in \mathbb{Q})$
↳ the proof is a classic example of a "non-constructive" proof: it shows such x, y exist without determining explicitly what they are.

PF: Consider $\sqrt{2}^{\sqrt{2}}$. Either $\sqrt{2}^{\sqrt{2}}$ is rational, or irrational. If $\sqrt{2}^{\sqrt{2}}$ is rational, let $x = \sqrt{2}$ and $y = \sqrt{2}$. Then x and y are both irrational but $x^y = \sqrt{2}^{\sqrt{2}}$ is rational, and the statement is proved.

- If $\sqrt{2}^{\sqrt{2}}$ is irrational, let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. Then x, y are both irrational but

$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2$$

which is rational, and the statement is proved.

- Since $\sqrt{2}^{\sqrt{2}}$ must be either rational or irrational, and in either case we can produce the desired x and y, the proposition holds. ✓

<u>Note</u>: - it turns out $\sqrt{2}^{\sqrt{2}}$ is <u>irrational</u> (but this is harder to prove).

<u>Indirect Proof Strategy</u>:

- Assume $\neg(\exists x \in S.\ P(x))$ and derive a contradiction
- That is, assume $\forall x \in S. \neg P(x)$ and derive a contradiction

Ex ③ Fix $n \in \mathbb{N}$ and suppose $a_1, a_2, \ldots, a_n \in \mathbb{R}$. Then at least one of $a_1, \ldots, a_n$ is as large as their average, that is,

$$\exists k \in [n].\ a_k \geq \frac{1}{n} \sum_{i=1}^{n} a_i$$

Pf. - Suppose not, towards a contradiction

- That is, suppose that for every $k \in [n]$ we have

$$a_k < \frac{1}{n} \sum_{i=1}^{n} a_i$$

- For simplicity let $S = \sum_{i=1}^{n} a_i$

Then our assumption is that for every $k \in [n]$ we have

$$a_k < \frac{S}{n}$$

Then we have

$$S = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$
$$< \frac{S}{n} + \frac{S}{n} + \cdots + \frac{S}{n},$$

(since we have assumed $a_k < \frac{S}{n}$ for every k)

$$= n \cdot \frac{S}{n}$$

$$= S.$$

- We have just argued that $S < S$, a contradiction.
- Thus our assumption was false, and the prop'n must be true. ✓

Universal Proofs

General Form: $\forall x \in S . P(x)$.

Direct Proof Strategy:
- Let $x \in S$ be arbitrary but fixed
- Prove $P(x)$ holds

Examples ① **Prop'n** $\forall x, y \in \mathbb{R} . xy \leq \left(\frac{x+y}{2}\right)^2$

Pf: Let $x, y \in \mathbb{R}$ be arbitrary and fixed.

Since squares are always non-negative we know

$$0 \le (x-y)^2$$

- Hence:

$$0 \le x^2 - 2xy + y^2$$

- Hence:

$$2xy \le x^2 + y^2$$

- We may add $2xy$ to both sides of the inequality to obtain:

$$4xy \le x^2 + 2xy + y^2$$

- Hence

$$4xy \le (x+y)^2$$

- i.e. $$xy \le \frac{(x+y)^2}{4}$$

- hence $$xy \le \left(\frac{x+y}{2}\right)^2$$ ✓

Since, $x, y \in \mathbb{R}$ were arbitrary, the prop'n is proved. ✓

<u>Note</u>: This prop'n is one version of the "AM GM inequality"

the <u>arithmetic</u> <u>mean</u> (AM) of x, y is $\frac{x+y}{2}$

- the geometric mean (GM) of x, y is $\sqrt{xy}$

- from the prop'n $\sqrt{xy} \leq \frac{x+y}{2}$ $(x, y \geq 0)$
- that is $GM \leq AM$.

Indirect Proof Strategy

Assume $\neg \forall x \in S.\ P(x)$
$(\text{i.e. } \exists x \in S.\ \neg P(x))$
and get a contradiction

Ex ② Prop'n $\sqrt{2}$ is irrational,
i.e. $\forall a, b \in \mathbb{Z}.\ \frac{a}{b} \neq \sqrt{2}$

Pf. - Suppose not, that is, suppose $\exists a, b \in \mathbb{Z}$ s.t.

$$\frac{a}{b} = \sqrt{2}$$

- We may assume $\frac{a}{b}$ is in reduced form, i.e. a and b have no common factors, since if not we can cancel these factors and get a fraction in reduced form.

– Now, since $\frac{a}{b} = \sqrt{2}$ we have

$$a = \sqrt{2}b$$
$$\Rightarrow \quad a^2 = 2b^2$$

Hence a^2 is even. It follows a must be even as well (Why? we'll prove this in a moment).

So $\exists k \in \mathbb{N}$ s.t. $a = 2k$.

So then $a^2 = 4k^2$

Combining this w/ the above:

$$2b^2 = 4k^2$$

Hence:

$$b^2 = 2k^2$$

Reasoning as before we see that b^2, and hence b, is even. But then <u>both</u> a, b are even, and therefore share a factor of 2.

– This is a contradiction, as we supposed a, b shared no common factors

– The prop'n follows. ✓

Conditional Claims

General Form: $P \Rightarrow Q$

Three strategies:

① Direct: Assume P holds, show Q holds

② Contrapositive: Show $\neg Q \Rightarrow \neg P$, i.e. assume $\neg Q$ holds, prove $\neg P$.

③ Indirect: Assume $\neg(P \Rightarrow Q)$, i.e. assume $P \wedge \neg Q$. Derive a contradiction.

Ex's ① (Direct) Let O denote the set of odd integers (not necessarily positive)

Then:

$$\forall n \in \mathbb{Z}\,(n \in O \Rightarrow n^2 - 1 \text{ is divisible by } 4)$$

(or, even more symbolically):

$$\forall n \in \mathbb{Z}\,(n \in O \Rightarrow \exists M \in \mathbb{Z}.\ n^2 - 1 = 4M)$$

Pf. Overall, this is a universal claim. So fix an arbitrary $n \in \mathbb{Z}$ and assume $n \in O$.

Then $\exists k \in \mathbb{Z}$ s.t. $n = 2k+1$

$\Rightarrow n^2 = (2k+1)^2 = 4k^2 + 4k + 1$

$\Rightarrow n^2 - 1 = 4k^2 + 4k$

$\Rightarrow n^2 - 1 = 4(k^2 + k)$

- Hence $\exists M \in \mathbb{Z}$ s.t. $n^2 - 1 = 4M$ (namely $M = k^2 + k$)
- Hence $n^2 - 1$ is divisible by 4. ✓

② (contrapositive) <u>Prop'n</u>
$\forall m, n \in \mathbb{Z}$ (if mn is even, then at least one of m, n is even)

i.e. $\forall m, n \in \mathbb{Z}. \ (mn \in E \Rightarrow (m \in E \vee n \in E))$

<u>Pf</u>: - Let $m, n \in \mathbb{Z}$ be arbitrary.
- Suppose that neither m nor n is even, i.e. that $m \notin E \wedge n \notin E$
- then for some $k, l \in \mathbb{Z}$ we have $m = 2k+1$ and $n = 2l+1$.
- hence

$$mn = (2k+1)(2l+1)$$
$$= 4kl + 2k + 2l + 1$$

$$= 2(2k\ell + k + \ell) + 1$$
$$= 2M + 1$$

where $M = 2k\ell + k + \ell$

- This shows mn is odd, i.e. $mn \notin E$.

- We have proved
$$m \notin E \wedge n \notin E \Rightarrow mn \notin E$$
i.e.
$$\neg(m \in E \vee n \in E) \Rightarrow \neg(mn \in E)$$

- By contrapositive we have
$$mn \in E \Rightarrow m \in E \vee n \in E$$

- Since m, n were arbitrary, the prop'n is proved.

(3) (Indirect) Prop'n

$\forall x \in \mathbb{R} (x > 0 \Rightarrow x + \frac{1}{x} \geq 2)$

Pf: - Fix an arbitrary $x \in \mathbb{R}$

- Suppose that $\underset{P}{x > 0}$ but $\underset{\neg Q}{x + \frac{1}{x} < 2}$

- Since x is positive we may multiply both sides of the inequality to obtain

$$x^2 + 1 < 2x$$

- Hence

$$x^2 - 2x + 1 < 0$$

- i.e.

$$(x-1)^2 < 0$$

But this is a contradiction since the square of a real number is always nonnegative

- Hence we have $x > 0 \Rightarrow x + \frac{1}{x} \geq 2$

- Since x was arbitrary, the prop'n is proved. ✓

Biconditional Claims

General Form: $P \Leftrightarrow Q$

Strategy: prove $P \Rightarrow Q$ and
$Q \Rightarrow P$

Ex Prop'n An integer is even if and only if its square is even, i.e.

$$\forall n \in \mathbb{Z} \ (n \in E \Leftrightarrow n^2 \in E)$$

Pf. - Fix $n \in \mathbb{Z}$ arbitrary.
- ($\Rightarrow$) assume $n \in E$.
  Then $n = 2K$ for some $K \in \mathbb{Z}$
  Then $n^2 = 4K^2$
  Hence $n^2 = 2M$ (where $M = 2K^2$)
  and we see $n^2 \in E$ ✓

- ($\Leftarrow$) To prove $n^2 \in E \Rightarrow n \in E$ we'll prove the contrapositive, i.e. $n \notin E \Rightarrow n^2 \notin E$
- So assume $n \notin E$
- Then for some $M \in \mathbb{Z}$ we have
  $n = 2M + 1$

- hence $n^2 = (2M+1)^2$
$= 4M^2 + 4M + 1$
$= 2(2M^2 + 2M) + 1$
$= 2N + 1$ (where $N = 2M^2 + 2M$)
- hence n^2 is odd
i.e. $n^2 \notin E$

- by contrapositive we have shown $n^2 \in E \Rightarrow n \in E$
- along with our previous argument that $n \in E \Rightarrow n^2 \in E$ we have $n \in E \Leftrightarrow n^2 \in E$

- Since $n \in \mathbb{Z}$ was arbitrary, the prop'n follows. ✓