Infinity

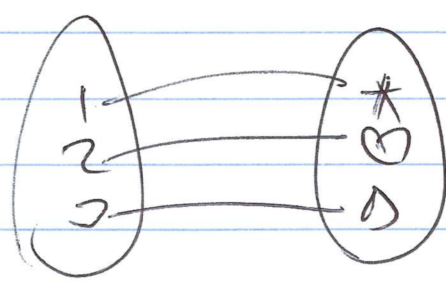
The concept of size

- we would say $\{*, \heartsuit, \Delta\}$ has <u>3 elements</u>, or is of <u>size 3</u>

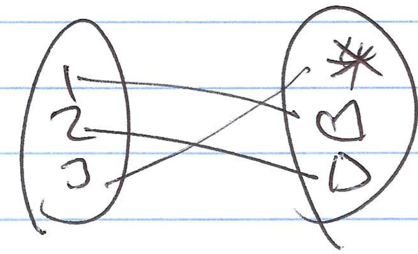
- why? By counting:

$$\{\underset{1}{*}, \underset{2}{\heartsuit}, \underset{3}{\Delta}\}$$

- in doing so we are implicitly defining a <u>bijection</u> between $\{*, \heartsuit, \Delta\}$ and $\{1, 2, 3\}$



- there is more than one way to do this:



– we will use "existence of a bijection between" as our concept of "of the same size."

Def'n Given sets A, B, we say A, B have the same cardinality and write $A \sim B$ iff $\exists f: A \to B$ a bijection.

Note: – in set theory courses one defining, for every set A, the cardinal number $|A|$ of A.
– then show $A \sim B$ iff $|A| = |B|$.
– defining cardinal numbers beyond our scope. For us, $|A| = |B|$ is just abbreviation for $A \sim B$.

Property of $\sim$:

① For any A, $id_A : A \to A$ is a bijection. So $A \sim A$, i.e. $\sim$ is reflexive.

② if $\exists$ a bijection $f: A \to B$ then $f^{-1}: B \to A$ is a bijection. So $A \sim B$ implies $B \sim A$, i.e. $\sim$ is symmetric

(3) On HW you show: if $f: A \to B$ and $g: B \to C$ are bijections then $g \circ f: A \to C$ is a bijection. So $A \sim B$ and $B \sim C \Rightarrow A \sim C$, i.e. $\sim$ is transitive

↪ $\sim$ is an equivalence relation on sets!

<u>Def'n</u> Let A, B be sets.

(1) We write $A \lesssim B$ (or $|A| \leq |B|$) iff $\exists$ an <u>injection</u> $f: A \to B$

(2) We write $A \gtrsim B$ (or $|A| \geq |B|$) iff $\exists$ a <u>surjection</u> $f: A \to B$

↪ we'll also write $A < B$ to mean $A \lesssim B$ and $A \not\sim B$.

<u>Notice</u>: - $A \gtrsim B$ is not "reverse of" $A \lesssim B$, i.e. is not asserting an injection from B to A.

- but this follows:

<u>Theorem</u> For all sets A, B we have

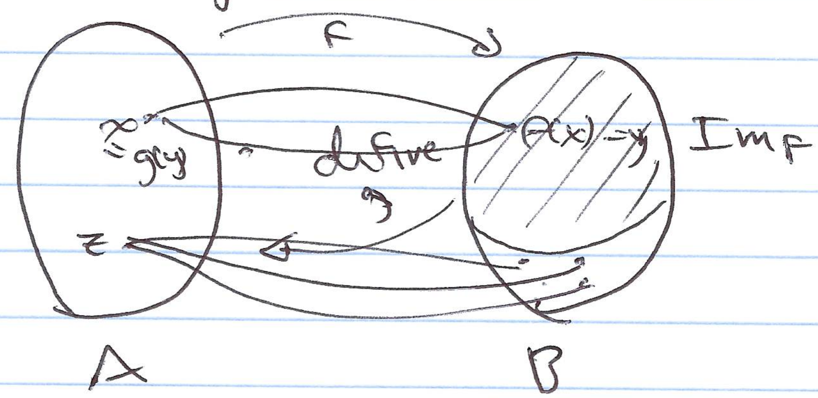
$$A \lesssim B \text{ iff } B \gtrsim A.$$

Pf: (Sketch)

($\Rightarrow$) - Suppose $A \lesssim B$

- then $\exists f: A \to B$ an <u>injection</u>
- want to construct $g: B \to A$ a <u>surjection</u>.

$\boxed{\text{Idea}}$ Let g be "inverse of f" on $Im_f(A)$, and for anything left over map to a single element $z \in A$.



Explicitly, fix a $z \in A$.

Define

$$g = \{(b,a) \mid (a,b) \in f\} \cup \{(b,z) \mid b \notin Im_f\}$$

this says:

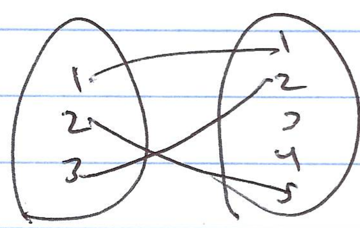
$$g(b) = \begin{cases} a & \text{if } f(a) = b \\ z & \text{if } b \notin Im_f \end{cases}$$

then g is clearly a surjection from B to A, i.e. $B \gtrsim A$. ✓

<u>ex</u>: let $f: [3] \to [5]$ be defined by

$f(1) = 1$
$f(2) = 5$
$f(3) = 2$

↳ f is an injection



then define $g: [5] \to [3]$ by

$g(1) = 1$
$g(2) = 3$
$g(5) = 2$

and $g(3) = g(4) = 1$.

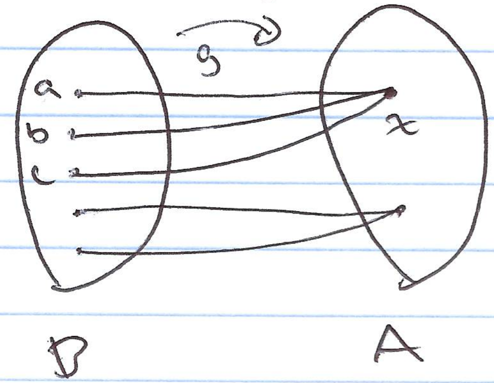
↳ g is a surjection ✓

($\Leftarrow$) Suppose $B \gtrsim A$.

- then there is a surjection $g: B \to A$
- want to construct an injection $f: A \to B$

$\boxed{\text{idea}}$ for every $x \in A$, pick an el't y_x in $\mathrm{PreIm}_g(\{x\})$

and let ~~g~~ $f(x) = y_x$

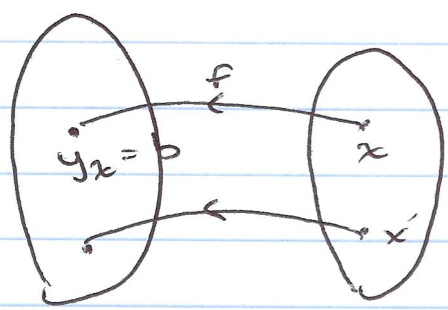


$g(a) = g(b) = g(c) = x$

~~let~~ pick one

say $y_x = b$

let $f(x) = y_x$

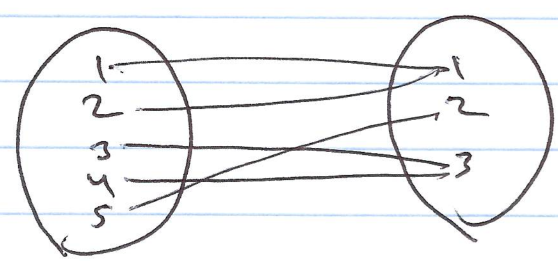


then $f : A \to B$ will be injective ✓

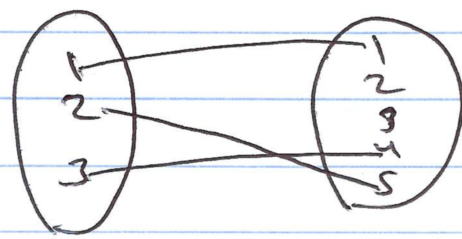
– hence $A \lesssim B$. ✓

e.g: consider the schematic for

$g : [5] \to [3]$ below



can define an injection $f: [3] \to [5]$



For infinite sets, to guarantee such an injection exists need to invoke the <u>axiom of choice</u>.

<u>Properties of $\lesssim$</u>

① $A \lesssim A$ and $A \gtrsim A$ since id_A is both an injection and surjection so these relations are reflexive.

② if we have injections $f: A \to B$ and $g: B \to C$ then $g \circ f: A \to C$ is an injection, so $A \lesssim B$ and $B \lesssim C \implies A \lesssim C$.

- Similarly $A \gtrsim B$, $B \gtrsim C \implies A \gtrsim C$.

- So relations are <u>transitive</u>

③ Are they antisymmetric?

Not literally: if $A \lesssim B$ and $B \lesssim C$ then needn't have $A = C$ (consider $\{1,2,3\}$ and $\{*, \square, \Delta\}$)

↳ but we'll show that in such a case $A \sim B$!

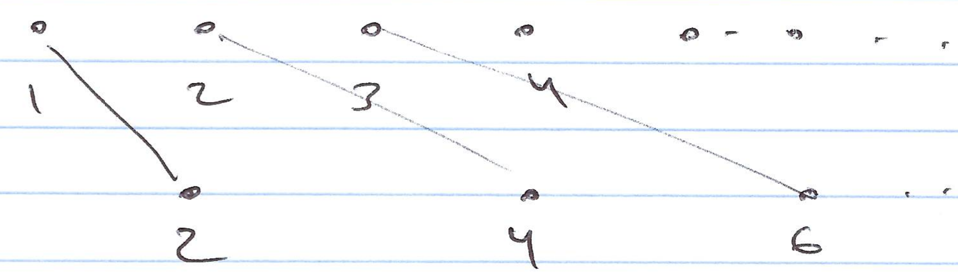
But first an interlude:

Paradoxes of infinity

① "There are as many even numbers as whole numbers"

i.e. $E \sim \mathbb{N}$

Proof: $f : \mathbb{N} \to E$ defined by $f(n) = 2n$ is a bijection.



② (Galileo) "There are as many perfect squares as whole numbers!"

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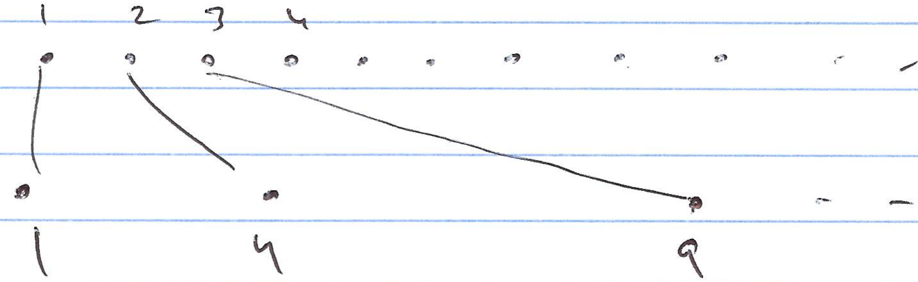
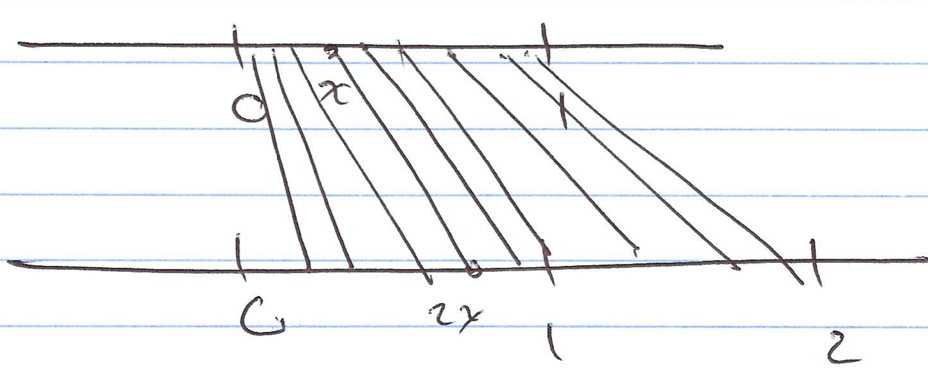
i.e. Let $S = \{1, 4, 9, 16, \ldots\}$
Then $S \sim \mathbb{N}$

<u>Pf</u>: $f: \mathbb{N} \to S$ defined by $f(n) = n^2$ is a bijection



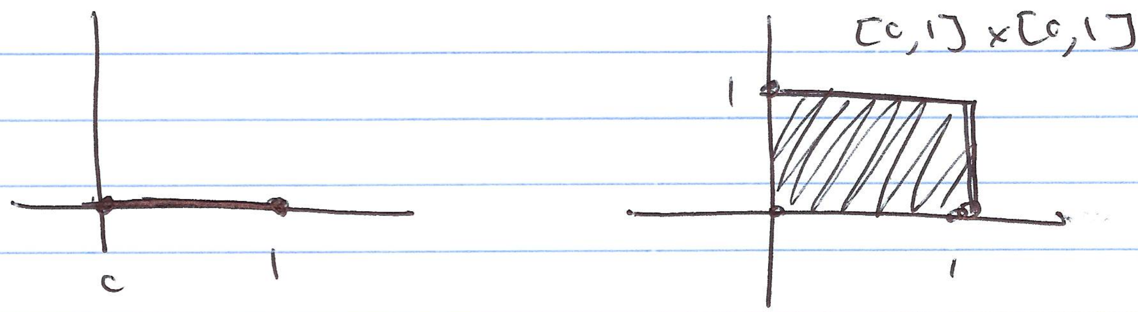
③ "there are as many points between 0 and 1 as between 0 and 2."

Let $A = [0,1]$
$B = [0,2]$
then $f: A \to B$
$f(x) = 2x$ is
a bijection



4 "The side is as large as the square"

~~Thm~~ there is a bijection $f: [0,1] \to [0,1] \times [0,1]$



Pf: beyond scope.

8 "There's always a vacancy at Hilbert's Hotel"

- Suppose there is a hotel with infinitely many rooms: #1, 2, 3, 4, ...
- Suppose all are filled and a new guest arrives — how can the staff accommodate her?
- Move everyone down a room and put her in room 1!

($f(n) = n+1$ is a bijection from $\mathbb{N}$ to $\mathbb{N} \setminus \{1\}$)

6 "A vanishing abundance"

- You find yourself next to an infinite pile of balls and a large bucket.
- At 11 am you take 10 balls from the pile, place them in the bucket, and then remove one ball from the bucket and put it aside (9 balls remain)
- At 11:30 you put 10 more in the bucket, and then remove one and put it aside (18 balls remain)
- At 11:45 you repeat
- And so on: ~~each time~~ whenever the time before noon halves, you repeat

How many balls are in the bucket at noon?

These examples illustrate the strangeness of ~~of~~ ~~size~~ the concept "the size of an infinite set"

<u>Def'n</u> - a set X is <u>finite</u> iff X is empty or $\exists n \in \mathbb{N}$ and a bijection $f: [n] \to X$

- a set is infinite iff it is not finite, that iff $X \neq \emptyset$ and $\forall n \in \mathbb{N}$ there is <u>no</u> bijection $f: X \to [n]$.

The set $\mathbb{N}$ is the <u>smallest</u> infinite set in the following sense.

<u>Theorem</u> if X is an infinite set then $\mathbb{N} \precsim X$.

<u>Pf</u>: - We define an injection $f: \mathbb{N} \to X$ "inductively"

- Since X is not empty $\exists x_1 \in X$.
- Let $f_1 = \{(1, x_1)\}$
- Since X is not finite, f_1 is not a bijection of $[1]$ with X. Hence $\exists x_2 \in X$, $x_2 \neq x_1$
- Let $f_2 = \{(1, x_1), (2, x_2)\}$

i.e. f_n an injection

- at stage n suppose we have defined $f_n = \{(1,x_1),(2,x_2),\dots,(n,x_n)\}$ s.t. $\forall i,j \le n$ if $i \ne j$ then $x_i \ne x_j$.
- then since X is not finite $f_n : [n] \to X$ is not a bijection
- so $\exists x_{n+1} \in X$ distinct from $x_1,\dots,x_n$
- let $f_{n+1} = \{(1,x_1),\dots,(n+1,x_{n+1})\}$

- By induction, then $\forall n$ we can define an injection $f_n : [n] \to X$
- let $f = \bigcup_{n \in \mathbb{N}} f_n$

$$= \{(1,x_1),(2,x_2),\dots\}$$

- then f is an injection $f : \mathbb{N} \to X$ ✓
- hence $\mathbb{N} \precsim X$.

- Thm says that among infinite sets, $\mathbb{N}$ is as small as possible
- hence any infinite sets that appear smaller (e.g. E, O, etc) are <u>not</u>
- But OTOH, $\mathbb{N}$ is very large in the sense that many sets X which seem larger than $\mathbb{N}$ are not.

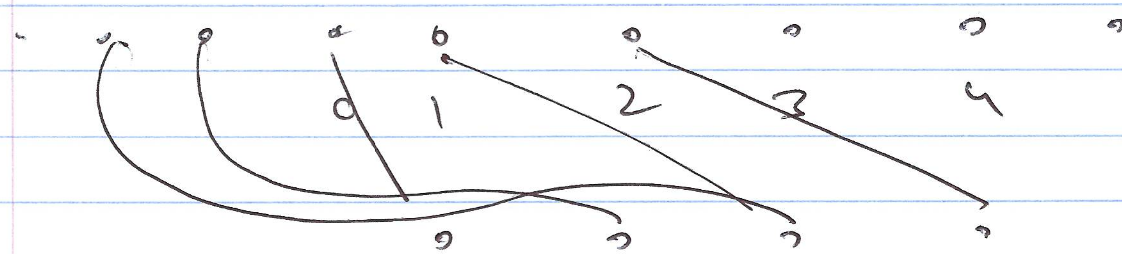
Def'n A set X is called countable iff $\mathbb{N} \sim X$.

Ex's ① $\mathbb{Z}$ is countable, i.e. $\mathbb{Z} \sim \mathbb{N}$.

Pf: We showed before that $f : \mathbb{Z} \to \mathbb{N}$ defined by

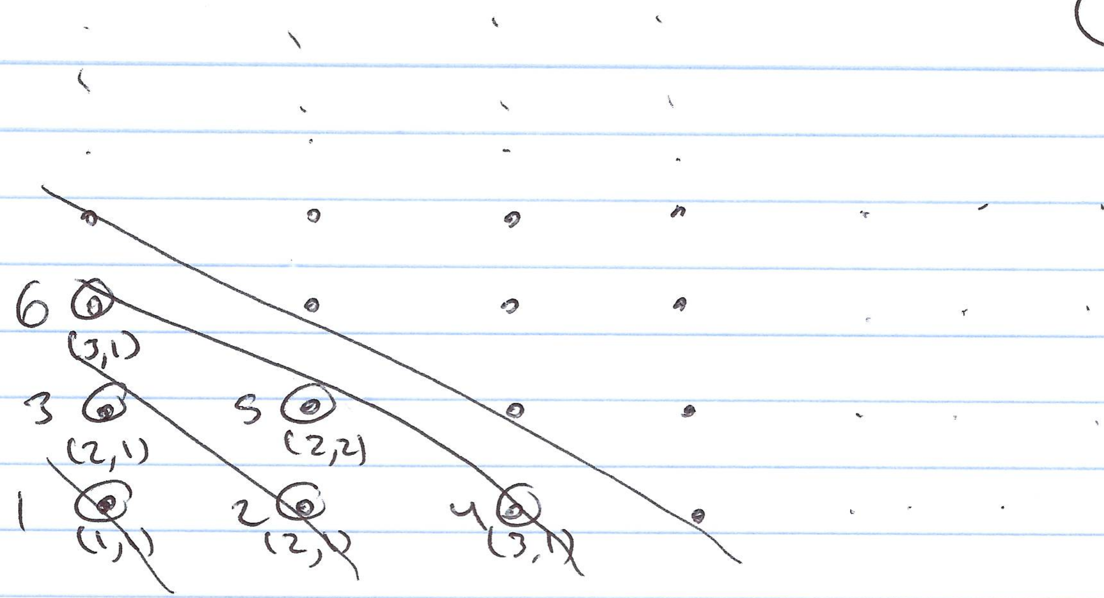
$$f(n) = \begin{cases} 2n & n > 0 \\ 2(-n)+1 & n \leq 0 \end{cases}$$

is a bijection



② $\mathbb{N} \times \mathbb{N}$ is countable, i.e. $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$

Pf by picture:



Continuing in this way ("cantor along diagonals") we get a bijection

$$f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$$

e.g. $f(1) = (1,1)$
$f(5) = (2,2)$
$f(6) = (3,1)$ etc. ✓

Another approach

2.b By a different route, we can show $\mathbb{N} \times \mathbb{N} \lesssim \mathbb{N}$.

Pf: Day 1 we proved there are infinitely many primes

- let $p_1, p_2, p_3, \ldots$

be an increasing enumeration

SKIP 16, 17, 18

of the primes

so e.g. $p_1 = 2$
$p_2 = 3$
$p_3 = 5$
$p_4 = 7$
$p_5 = 11$
etc...

<u>Define</u>: $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by
$f(n,m) = p_n^m$

so e.g. $f(1,3) = p_1^3 = 2^3 = 8$
$f(2,2) = p_2^2 = 3^2 = 9$
$f(5,2) = p_5^2 = 11^2 = 121$
etc...

<u>Claim</u> f is an injection

Pf:

- Fix $(n,m), (k,\ell) \in \mathbb{N} \times \mathbb{N}$
- Suppose $(n,m) \neq (k,\ell)$

<u>Case 1</u>: $n \neq k$

- then $f(n,m) = p_n^m \neq p_k^\ell = f(k,\ell)$

different primes

(technically relies on FTOA, we'll prove that later)

Case 2: $n = k$ but $m \neq l$.
Then $f(n,m) = p_n^m \neq p_n^l$
different powers

$= p_k^l = f(k,l)$

In either case $f(n,m) \neq f(k,l)$
So f is injective

Hence $\mathbb{N} \times \mathbb{N} \precsim \mathbb{N}$
we knew $\mathbb{N} \precsim \mathbb{N} \times \mathbb{N}$ → by Thm or
$g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$
$g(n) = (n,1)$

From this alone can we
conclude $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$?

We need:

<u>Theorem</u> (Cantor - Schroeder - Bernstein)
For any sets A, B, if $A \precsim B$ and
$B \precsim A$ then $A \sim B$.

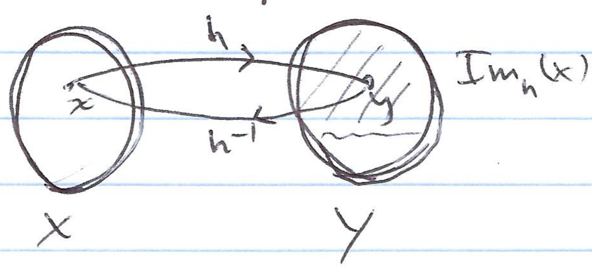
Pf: Fix sets A, B and suppose
there are injections
$f : A \to B$
$g : B \to A$

- we need to construct a <u>bijection</u> $F: A \to B$.

- if either f or g is a bijection to begin with we are done.
- so assume neither are surjective

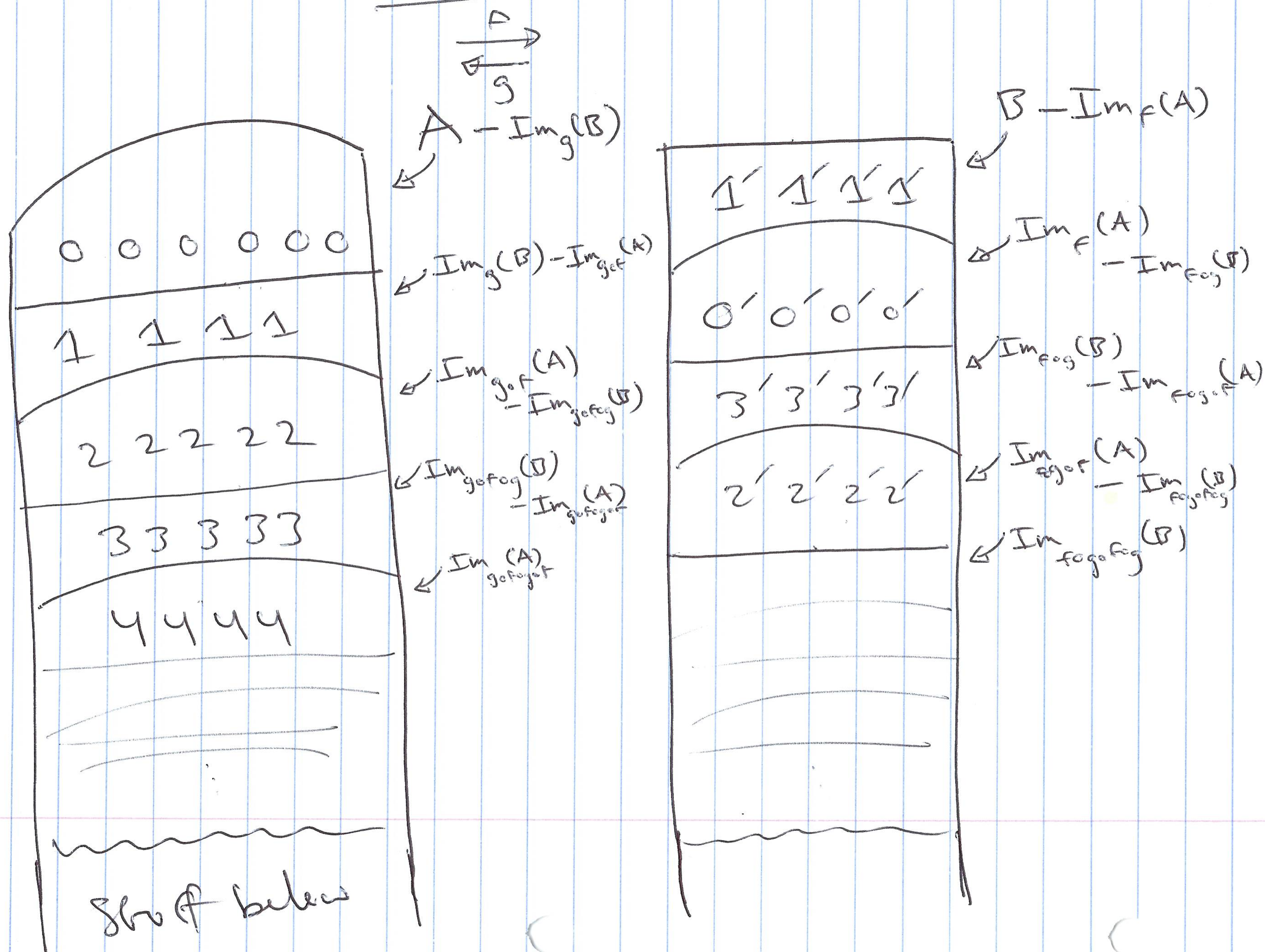
$\boxed{\text{Side note}}$ - In general if $h: X \to Y$ is an injection, then h is a bijection from X to $Im_h(X)$.

- if $y \in Im_h(X)$ we define $h^{-1}(y) =$ unique x s.t. $h(x) = y$.



Back to proof:

THE Picture


f
g
A – Im_g(B)
Im_g(B) – Im_{gof}(A)
Im_{gof}(A) – Im_{gofog}(B)
Im_{gofog}(B) – Im_{gofogof}(A)
Im_{gofogof}(A)
0 0 0 0 0 0
1 1 1 1
2 2 2 2 2
3 3 3 3 3
4 4 4 4
Stuff below
B – Im_f(A)
Im_f(A) – Im_{fog}(B)
Im_{fog}(B) – Im_{fogof}(A)
Im_{fogof}(A) – Im_{fogofog}(B)
Im_{fogofog}(B)
1′ 1′ 1′ 1′
0′ 0′ 0′ 0′
3′ 3′ 3′ 3′
2′ 2′ 2′ 2′

Let $X = (A - \mathrm{Im}_g(B)) \cup (\mathrm{Im}_{g\circ f}(A) - \mathrm{Im}_{g\circ f\circ g}(B))$

$\cup (\mathrm{Im}_{g\circ f\circ g\circ f}(A) - \mathrm{Im}_{g\circ f\circ g\circ f\circ g}(B)) \cup \cdots$

$=$ (region 0) $\cup$ (region 2) $\cup$ (region 4) $\cup \cdots$

Let Y = everything else in A

$=$ (region 1) $\cup$ (region 3) $\cup \cdots$
$\cup$ (stuff below)

Define $F : A \to B$

$$F(x) = \begin{cases} f(x) & \text{if } x \in X \\ g^{-1}(x) & \text{if } x \in Y \end{cases}$$

<u>Claim</u> F is a bijection

<u>Pf (Sketch)</u> (i) <u>injectivity</u>: Fix $x, y \in A$

suppose $F(x) = F(y)$

<u>Case 1</u>: $x \in X$, $y \in X$

Then $F(x) = f(x)$
$F(y) = f(y)$

so injectivity of $f \Rightarrow x = y$

<u>Case 2</u>: $x \in Y$, $y \in Y$

then $F(x) = g^{-1}(x)$
$F(y) = g^{-1}(y)$

Since g is injective, g^{-1} is injective where its defined, so $\Rightarrow x = y$

<u>Case 3</u> $x \in X$, $y \in Y$

Then x is in an even region n
y is in an odd region m
(or y below)

then $F(x)$ is in an even region n'
and $F(y)$ is in an odd region m'
(or $F(y)$ below)

↳ contradiction as these regions are disjoint and we assumed $F(x) = F(y)$.

<u>Case 4</u> $x \in Y$ and $y \in X$
Similar to Case 3.

↳ so F is injective ✓

<u>Surjectivity</u> (even sketchier)
Fix $y \in B$.

<u>Case 1</u>: - y is in an even region n'
- then there is $x \in$ region $n \subseteq X$
s.t. $F(x) = y$
hence $F(x) = y$

Case 2: - y is in an odd region m
- ~~then then~~ $x = g(y) \in m$
- hence $F(x) = g^{-1}(x) = g^{-1}(g(x)) = y$

Case 3: - y is "below in Y"
Believe me: there is x "below in X"
s.t. $g^{-1}(x) = F(x) = y$.

↳ so F is surjective. ✓

- CBS can be a very powerful tool in proving equivalence of sets
- shows that while $\lesssim$ is not literally antisymmetric, is antisymmetric up to $\sim$.
i.e. $A \lesssim B$, $B \lesssim A \Rightarrow A \sim B$

In "cardinality notation"

$$|A| \leq B \text{ and } |B| \leq |A| \Rightarrow |A| = |B|$$

↳ ~~[illegible]~~

① - we showed above $N \times N \lesssim N$ and we knew $N \lesssim N \times N$
- hence by CBS $N \times N \sim N$

even without producing an explicit bijection $f: \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ ourselves.

(2) We can generalize the above to prove:

<u>Theorem</u>: if A, B are countable sets then $A \times B$ is countable.

<u>Pf</u>: - Suppose $A \sim \mathbb{N}$ and $B \sim \mathbb{N}$.
- then there are bijections
$$f: A \to \mathbb{N}$$
$$g: B \to \mathbb{N}$$
- define
$$F: A\times B \to \mathbb{N}\times\mathbb{N}$$
by $F(a,b) = (f(a), g(b))$

then F is a bijection

<u>Pf</u>: you try.

(3) Combing (1) and (2) we can prove that in fact the set of rationals $\mathbb{Q}$ is countable (amazing!)

<u>Prop'n</u> $\mathbb{N} \sim \mathbb{Q}$

<u>Pf:</u> - Clearly $\mathbb{N} \lesssim \mathbb{Q}$ since $f: \mathbb{N} \to \mathbb{Q}$ defined by $f(n) = n$ is an injection

- above we proved $\mathbb{Z} \sim \mathbb{N}$
- hence by ② $\mathbb{Z} \times \mathbb{N} \sim \mathbb{N}$.

- we now claim:

<u>Claim:</u> $\mathbb{Z} \times \mathbb{N} \gtrsim \mathbb{Q}$.

<u>Pf:</u> - Consider the map $F: \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ defined by $F(m,n) = \frac{m}{n}$.

- this map is <u>not</u> injective e.g. $F(1,2) = F(3,6) = \frac{1}{2}$
- but it is clearly <u>surjective</u>
- if $q \in \mathbb{Q}$ and $q = \frac{m}{n}$ then $F(m,n) = q$.
- So $\mathbb{Z} \times \mathbb{N} \gtrsim \mathbb{Q}$ as claimed

↳ then $\mathbb{Q} \lesssim \mathbb{Z} \times \mathbb{N} \sim \mathbb{N}$
hence $\mathbb{Q} \lesssim \mathbb{N}$

↳ hence by CBS $\mathbb{N} \sim \mathbb{Q}$!

④ **Theorem** If $A_1, A_2, A_3, \ldots$ is a sequence of sets indexed by $\mathbb{N}$, and $\forall n \in \mathbb{N}$ we have $A_n \sim \mathbb{N}$, then

$$\bigcup_{n \in \mathbb{N}} A_n \sim \mathbb{N}$$

("a countable union of countable sets is countable")

Pf - Since each A_n is ctbl, we have, $\forall n \in \mathbb{N}$, a bijection

$$f_n : \mathbb{N} \to A_n$$

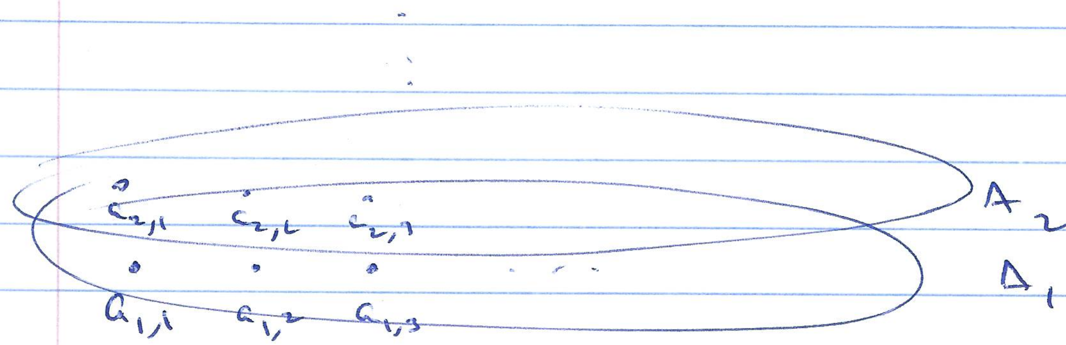
- Can think of f_n as giving us a way of listing A_n:
  - Let $a_{n,m}$ denote $f_n(m)$
  - Then:

$$\begin{array}{llll} A_n: & f_n(1) & f_n(2) & f_n(3) \ldots \\ & a_{n,1} & a_{n,2} & a_{n,3} \end{array}$$

Now define $F : \mathbb{N} \times \mathbb{N} \to \bigcup_{n \in \mathbb{N}} A_n$ by $F(n,m)$

$$= f_n(m) = a_{n,m}$$

Picture:



- Then F is a surjection.
- Fix $x \in \bigcup_{n \in \mathbb{N}} A_n$

- Suppose $x \in A_n$
- then $x = f_n(m)$ for some m
- so $x = F(n,m)$ ✓

↳ Hence $\mathbb{N} \times \mathbb{N} \gtrsim \bigcup A_n$

↳ Hence $\bigcup A_n \sim \mathbb{N}$ ✓

- So many examples of sets X that "appear larger" than $\mathbb{N}$ but are ctbl

- Q: are there any infinite sets X which are <u>not</u> countable?

- Yes!

<u>Theorem</u> (Cantor): $\mathbb{N} < P(\mathbb{N})$.
That is: $\mathbb{N} \lesssim P(\mathbb{N})$ but $\mathbb{N} \not\approx P(\mathbb{N})$

<u>Pf</u>: - the proof is by "magic trick" of diagonalization
- we'll show: if $f: \mathbb{N} \to P(\mathbb{N})$ is <u>any</u> function, then there is a set $T \in P(\mathbb{N})$ that is not in Img
- hence f is not a surjection

<u>Claim</u>: Let $f: \mathbb{N} \to P(\mathbb{N})$ be a function. Then f is not a surjection

<u>Pf</u>: Let $T = \{n \in \mathbb{N} \mid n \notin f(n)\}$

Illustration:
Suppose e.g. $f(1) = \{1, 7, 10\}$
$f(2) = \{1, 3, 5, 7, \ldots\}$
$f(3) = \emptyset$
$f(4) = \{2, 4, 6, \ldots\}$
⋮

Then. $1 \notin T$ because $1 \in f(1)$
$2 \in T$ because $2 \notin f(2)$
$3 \in T$ because $3 \notin f(3)$
$4 \notin T$ because $4 \in f(4)$
$\vdots$

So $T = \{2, 3, \ldots\}$

Claim 1 $(\forall n \in \mathbb{N})$ $f(n) \neq T$

Pf: Fix $n \in \mathbb{N}$. - if $n \in T$ then $n \notin f(n)$ by def'n of T.
- hence $T \neq f(n)$
- if $n \notin T$, then $n \in f(n)$, hence $T \neq f(n)$ as well.
- Since n was arbitrary, claim 1 follows

- hence f is not a surjection and original claim follows

- hence there is no bijection $f: \mathbb{N} \to P(\mathbb{N})$
i.e. $\mathbb{N} \not\approx P(\mathbb{N})$
- Since $\mathbb{N} \lesssim P(\mathbb{N})$ we have $\mathbb{N} < P(\mathbb{N})$ ✓

The same proof works in general:

<u>Theorem</u> For any set A, there is no surjection $f: A \to P(A)$.

Pf: Fix $f: A \to P(A)$

Let $T = \{ a \in A \mid a \notin f(a) \}$

Then $(\forall a \in A)\ f(a) \neq T$.

— Since $f: A \to P(A)$ defined by

$$f(a) = \{a\}$$

is an injection we always have $A \precsim P(A)$

— So Theorem above says $A < P(A)$.

→ It follows that there are <u>infinitely</u> <u>many</u> <u>levels</u> of <u>infinity</u>

$$\mathbb{N} < P(\mathbb{N}) < P(P(\mathbb{N})) < \cdots$$

Def'n if X is infinite and $\mathbb{N} \not\approx X$ then X is called uncountable.

↳ observe: $\mathcal{P}(\mathbb{N})$ is uncountable.
- what about $\mathbb{R}$?

Theorem $\mathbb{R}$ is uncountable.

Pf: - we'll actually argue that a seemingly "small" subset of $\mathbb{R}$ is uncountable.

- Let $R = \{x \in (0,1) \mid$ decimal expansion of x contains only 0's and 1's$\}$

- e.g.

$.1100111010 1\ldots \in R$
$.10101010\ldots \in R$
$.1111111\ldots \in R$
$.123456\ldots \notin R$
$1.101010 \ldots \notin R$.

- For $x \in R$, define x_n to be nth digit in decimal expansion of x

$x = .x_1 x_2 x_3 \ldots$

So e.g. if $x = .0010110\ldots$

then $x_1 = 0$

$x_2 = 0$

$x_3 = 1$ etc.

Claim Let $f: \mathbb{N} \to \mathbb{R}$ be any function. Then f is not a surjection.

Pf: Fix $f: \mathbb{N} \to \mathbb{R}$.

We define $r \in \mathbb{R}$ as follows

$r_n = 0$ iff $f(n)_n = 1$

$r_n = 1$ iff $f(n)_n = 0$

Picture:

Suppose $f(1) = .100101\ldots$

$f(2) = .0100111\ldots$

$f(3) = .1111000\ldots$

$f(4) = .00100011\ldots$

then $r_1 = 0$ $r_2 = 0$ $r_3 = 0$

$r_4 = 1$ etc

$r = .0001\ldots$

- Observe that $\forall n \in \mathbb{N}$
  $r \neq f(n)$
- why: $r_n \neq f(n)_n$
- hence f is not a surjection, as claimed.
- hence $\mathbb{N} < \mathbb{R} \lesssim \mathbb{R}$
- hence $\mathbb{N} < \mathbb{R}$. ✓

Sets of functions

- Consider the set F of functions $f : \mathbb{N} \to \{0,1\}$

$$F = \{ f \subseteq \mathbb{N} \times \{0,1\} \mid f : \mathbb{N} \to \{0,1\} \text{ is a function} \}$$

- we can think of a given $f \in F$ as an infinite 01-sequence

- e.g.
$$\begin{array}{l} \scriptstyle 1\,2\,3\,4 \\ 00110101\ldots = f \end{array}$$

means
$f(1) = 0$
$f(2) = 0$
$f(3) = 1$
$f(4) = 1$
$f(5) = 0$
$\vdots$
etc.

Theorem F is uncountable.

Pf: Diagonalize!

Claim: - suppose $g : \mathbb{N} \to F$ is a function.
- then g is not a surjection

Pf: define a function

$f : \mathbb{N} \to \{0,1\}$ as follows

$$f(n) = \begin{cases} 1 & \text{iff } g(n)(n) = 0 \\ 0 & \text{iff } g(n)(n) = 1 \end{cases}$$

then $\forall n \in \mathbb{N}$ $f \neq g(n)$. The claim follows

$g(1)(1)$

$g(1) = 0\,1\,0\,1\,0\,1 \ldots$

$g(2)(2)$

$g(2) = 0\,0\,1\,1\,0\,1\,0\,1\,1 \ldots$

$g(3) = 1\,1\,1\,1\,1\,1 \ldots$

$g(4) = 0\,0\,0\,1\,1\,1\,1 \ldots$

$f = 1\,1\,0\,0 \ldots$