Functions

↳ functions, like relations, ubiquitous in math

↳ but what "are" functions?

↳ intuitive defn: a rule that assigns to each x in a domain A a <u>unique</u> <u>output</u> $f(x)$ in codomain B

↳ can define them rigorously as a special type of relation

<u>Def'n</u> a function (with <u>domain</u> A and <u>codomain</u> B) is a relation $f \subseteq A \times B$ such that

for every $a \in A$, there is a unique $b \in B$ s.t. $(a,b) \in f$

(can write:

$$(\forall a \in A)\left((\exists b \in B)\,(a,b) \in f \;\wedge\; (\forall c \in B)\left[(a,c) \in f \Rightarrow c = b\right]\right)$$

↳ we write

$f: A \to B$ to indicate

$f \subseteq A \times B$ is a function

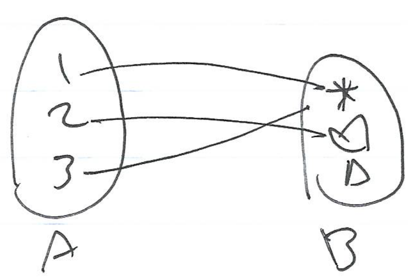
- will use standard notation $f(a) = b$ to mean $(a,b) \in f$.

<u>Note</u>: - def'n says every $a \in A$ is assigned an output $f(a) \in B$

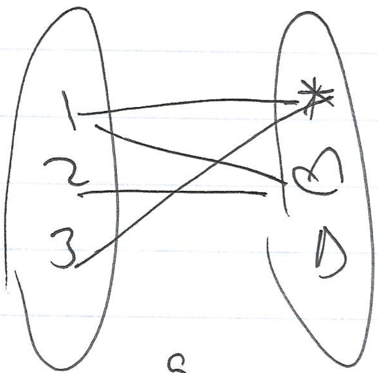
- does not insist for every $b \in B$ there is $a \in A$ s.t. $f(a) = b$ (functions w/ this property are called <u>onto</u> or <u>surjective</u>).

<u>ex</u>: ① let $A = \{1,2,3\}$
$B = \{*, ♡, \Delta\}$

then $f = \{(1,*), (2,♡), (3,*)\}$ is a function from A to B

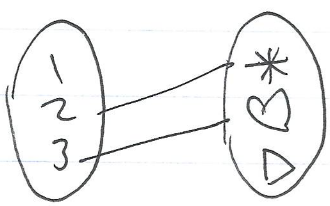


but $g = \{(1,*), (1,♡), (2,*), (3,*)\}$ is <u>not</u> (1 does not have a unique output)



g

nor is $h = \{(2,*), (3,\heartsuit)\}$, since 1 is not assigned an output



(2) we'll often define functions by some rule, as is common practice
e.g.

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) = x^2$$

or

$$g : \mathbb{R} \to \mathbb{Z}$$
$$g(x) = \lfloor x \rfloor$$

but behind the scenes still consider f, g to be <u>sets</u>

of ordered pairs (e.g. $(2,4) \in f$
$(\pi, 3) \in g$)

↳ one issue that arises when defining functions by rules is that sometimes such rules do not yield well-defined functions

e.g. Suppose we define $f: \mathbb{Q} \to \mathbb{Z}$ by the rule $f(\frac{m}{n}) = m+n$.

then this "function" is not one:

$$f(1/2) = 1+2 = 3 \neq 6 = 2+4 = f(\tfrac{2}{4})$$

but $\frac{1}{2} = \frac{2}{4}$ ...

- so f assigns multiple outputs to same input
- what's really going on here is that there's an implicit equiv. relation on fractions $(\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \ldots)$
- but we're defining f on ~~single~~ representatives of the

equiv. classes.

- in general, when given a rule defining some relation $f \subseteq A \times B$ to check f is a function one must verify:

  ① $\forall a \in A$ there is a $b \in B$ s.t. $(a,b) \in f$

  ② if $a = a'$ then $f(a) = f(a')$.

Equality of functions

Q: - what does it mean for functions $f: A \to B$ and $g: A \to B$ to be equal?

~~some~~ well, we've defined f, g as sets of ordered pairs, i.e. $f, g \subseteq A \times B$, so they're equal if they're equal as sets.

i.e.

$f = g$ iff ($f \subseteq g$ and $g \subseteq f$)

i.e. ($(a,b) \in f$ iff $(a,b) \in g$)

However in practice it's easier to use the following criterion:

<u>Theorem</u> if $f: A \to B$ and $g: A \to B$ are functions, then $f = g$ if and only if for every $a \in A$, $f(a) = g(a)$

<u>Pf</u>: exercise

Main point: functions may be equal despite being defined by different rules.

<u>ex</u>: Let $A = \{1, 2, 3\}$
define $f: A \to \mathbb{N}$
and $g: A \to \mathbb{N}$
by $f(x) = x^3 + 11x$
$g(x) = 6x^2 + 6$

Then $f(1) = 12$ $\quad$ $f(2) = 30$
$g(1) = 12$ $\quad$ $g(2) = 30$

$f(3) = 60$
$g(3) = 60$

here $f = g$!

anyone see <u>why</u> $f = g$ on A?

Images

though a function f need not "hit" every value in its codomain we do have a name for the set of outputs:

Def'n Suppose $f: A \to B$ is a function and $X \subseteq A$.

The image of X under f, denoted $Im_f(X)$ is defined as:

$$Im_f(X) = \{ b \in B \mid (\exists a \in X)\, f(a) = b \}$$

or more informally

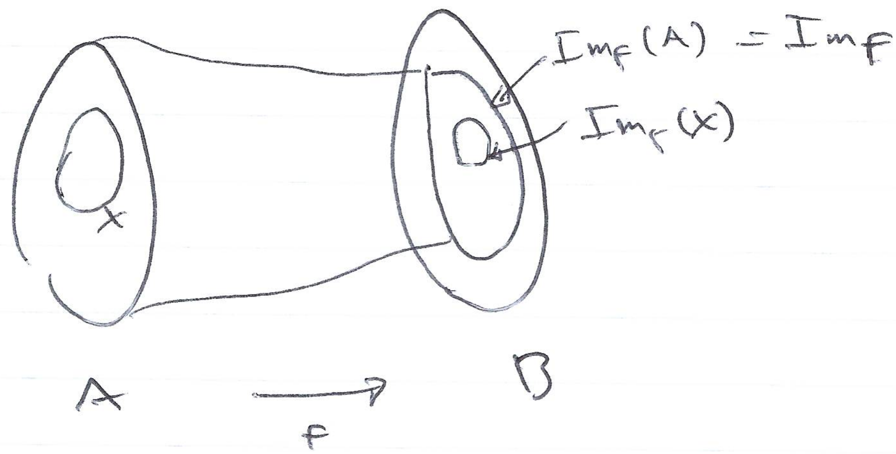
$$= \{ f(a) \mid a \in X \}$$

When $X = A$ we say simply that $Im_f(A)$ is the image of f and sometimes just write Im_f.

Def'n says: $-\, Im_f(X)$ is "set of outputs of el'ts in X"

$-\, Im_f = Im_f(A)$ is "set of all outputs"

Picture



ex: ① $A = \{1,2,3\}$ $\quad B = \{*, \heartsuit, \Delta\}$

$f = \{(1,*), (2,\heartsuit), (3,*)\}$

Then: $Im_f(\{1,3\}) = \{f(1), f(3)\}$

$= \{*, *\}$

$= \{*\}$

$Im_f = Im_f(A) = \{f(1), f(2), f(3)\}$

$= \{*, \heartsuit\}$

② Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$

Then: $Im_f(\{-1,0,1\}) = \{0,1\}$

$Im_f = \{x \in \mathbb{R} \mid x \geq 0\}$

- Functions add a layer of complexity to the basic set theory of $\cap$, $\cup$, etc... we studied earlier.
- e.g.:

Prop'n: Suppose $f: A \to B$ is a function and $S, T \subseteq A$.
Then:
$$\mathrm{Im}_f(S \cap T) \subseteq \mathrm{Im}_f(S) \cap \mathrm{Im}_f(T)$$

Pf: - fix $y \in \mathrm{Im}_f(S \cap T)$
- then $\exists x \in S \cap T$ s.t. $f(x) = y$
- hence $x \in S$ and $x \in T$
- hence $f(x) \in \mathrm{Im}_f(S)$ and $f(x) \in \mathrm{Im}_f(T)$

i.e. $y \in \mathrm{Im}_f(S)$ and $y \in \mathrm{Im}_f(T)$

i.e. $y \in \mathrm{Im}_f(S) \cap \mathrm{Im}_f(T)$

Since y was arbitrary the theorem is proved.

In general the containment cannot be reversed.

- e.g. Consider $f(x) = x^2$ on $\mathbb{R}$.
- let $S = \{-1, 0\}$
  $T = \{0, 1, 2\}$
- So $S \cap T = \{0\}$

Then: $Im_f(S) = \{f(-1), f(0)\}$
$= \{1, 0\}$

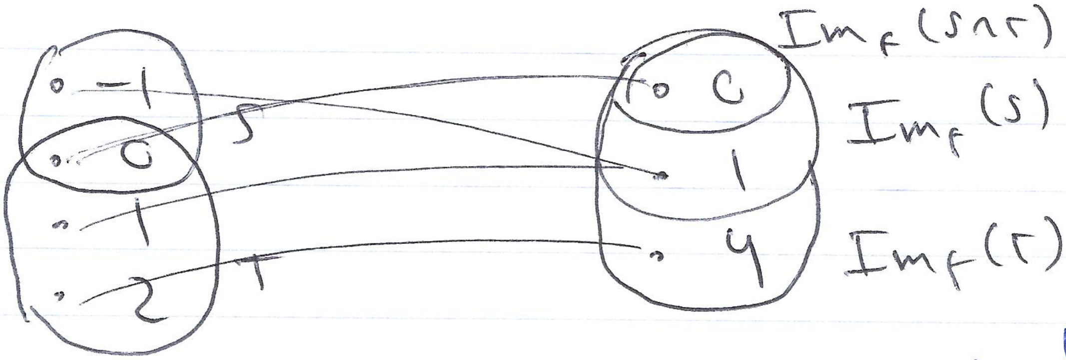
$Im_f(T) = \{f(0), f(1), f(2)\}$
$= \{0, 1, 4\}$

So $Im_f(S) \cap Im_f(T) = \{0, 1\}$

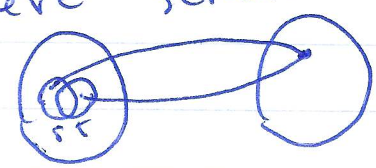
But $Im_f(S \cap T) = Im_f(\{0\})$
$= \{f(0)\}$
$= \{0\}$

So $Im_f(S \cap T) \neq Im_f(S) \cap Im_f(T)$

- fact that we don't have equality in above prop'n expresses an essential feature of functions: multiple inputs can have the same output



more schematically:



Preimages

Def'n Suppose $f: A \to B$ is a function and $Y \subseteq B$. Then the preimage of Y under f is defined as:

$$PreIm_f(Y) = \{x \in A \mid f(x) \in Y\}$$

Note: since $f(x) \in B$ for every $x \in A$, we don't separately define $PreIm_f(B)$, since this always $= A$.

ex: ① $A = \{1,2,3\}$

$B = \{*, \heartsuit, \Delta\}$

$f = \{(1,*), (2,\heartsuit), (3,*)\}$

then: $\mathrm{PreIm}_f(\{*\})$

$= \{x \in A \mid f(x) \in \{*\}\}$

$= \{x \in A \mid f(x) = *\}$

$= \{1,3\}$

$\mathrm{PreIm}_f(\{*,\heartsuit\})$

$= \{x \in A \mid f(x) \in \{*,\heartsuit\}\}$

$= \{1,2,3\}$

$\mathrm{PreIm}_f(\{\Delta\})$

$= \emptyset$.

note: when computing $\mathrm{PreIm}_f(Y)$ needn't be that every $y \in Y$ is in image of A!

② let $f: \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$.

then: $\mathrm{PreIm}_f(\{0,1\})$

$= \{x \in \mathbb{R} \mid f(x) \in \{0,1\}\}$

$= \{x \in \mathbb{R} \mid x^2 \in \{0,1\}\}$

$= \{-1, 0, 1\}$

$\mathrm{PreIm}_f([0,2])$

$= \{x \in \mathbb{R} \mid x^2 \in [0,2]\}$

$= \{x \in \mathbb{R} \mid 0 \le x^2 \le 2\}$

$= \{x \in \mathbb{R} \mid x^2 \le 2\}$

$= \{x \in \mathbb{R} \mid -\sqrt{2} \le x \le \sqrt{2}\}$

$= [-\sqrt{2}, \sqrt{2}]$.

$\mathrm{PreIm}_f([0,\infty))$

$= \{x \in \mathbb{R} \mid x^2 \in [0,\infty)\}$

$= \{x \in \mathbb{R} \mid x^2 \ge 0\}$

$= \mathbb{R}$

$\mathrm{PreIm}_f(\mathbb{R}) = \mathbb{R}$.

Can play with Images and PreImages.

<u>Prop'n</u> Suppose $f : A \to B$ is a function.

(i) Fix $X \subseteq A$.
Then $\mathrm{PreIm}_f(\mathrm{Im}_f(X)) \supseteq X$

(ii) Fix $Y \subseteq B$
Then $\mathrm{Im}_f(\mathrm{PreIm}_f(Y)) \subseteq Y$

Pf: (i) - fix $x \in X$
- let $y = f(x)$
- so $y \in Im_f(X)$
- so $\exists z \in Im_f(X)$ s.t. $f(z) = x$ namely $z = y$
- hence $x \in PreIm_f(Im_f(X))$

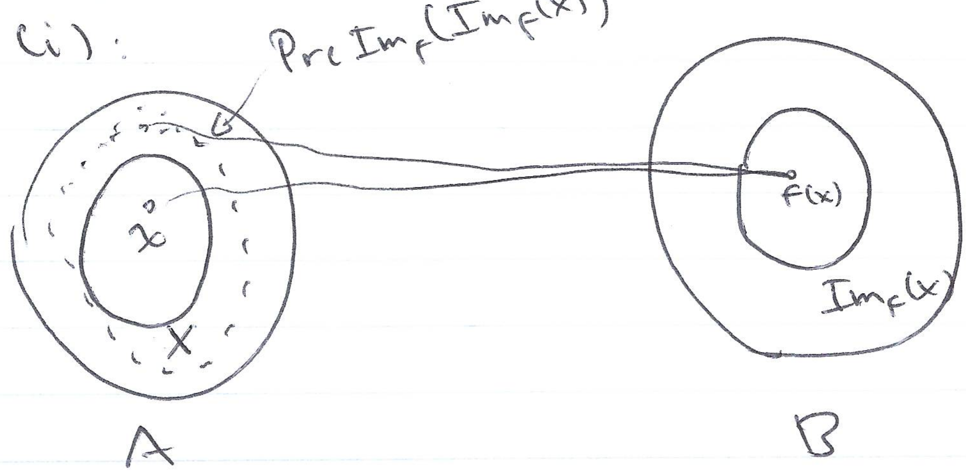
↳ since x was arbitrary, statement is proved.

(ii) - Fix $y \in Im_f(PreIm_f(Y))$
- then $\exists z \in PreIm_f(Y)$ s.t. $f(z) = y$
- but $f(z) \in \cancel{Im_f(PreIm_f(Y))}$ Y
- i.e. $y \in \cancel{Im_f(PreIm_f(Y))}$ Y

↳ since y was arbitrary statement is proved.

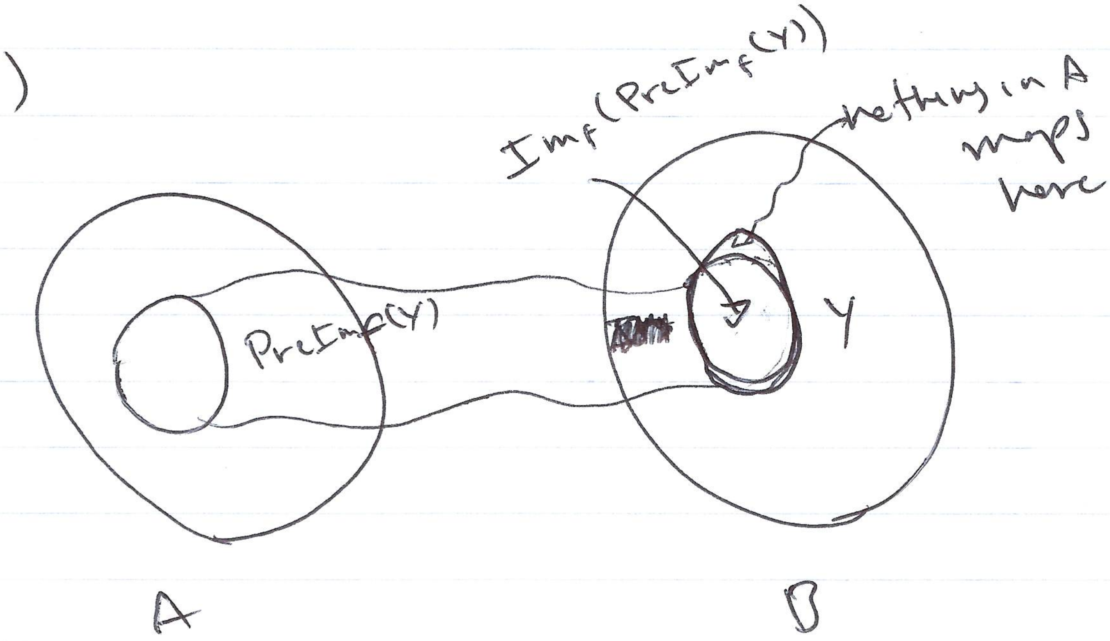
Picture:

(i):



(47)

(ii)



Neither containment can be reversed.

e.g. - Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$
- Let $X = \{1\}$

Then: $Im_f(X) = Im_f(\{1\})$

$= \{f(1)\}$
$= \{1\}$

$PreIm_f(Im_f(X)) = PreIm_f(\{1\})$
$= \{-1, 1\}$

So: $X \subsetneq \text{PreIm}_f(\text{Im}_f(X))$

Now: - Let $Y = \{-5, 1\}$

- Then $\text{PreIm}_f(Y)$

$$
\begin{aligned}
&= \text{PreIm}_f(\{-5,1\}) \\
&= \{x \in \mathbb{R} \mid f(x) \in \{-5,1\}\} \\
&= \{x \in \mathbb{R} \mid x^2 \in \{-5,1\}\} \\
&= \{-1, 1\}
\end{aligned}
$$

So: $\text{Im}_f(\text{PreIm}_f(Y))$

$$
\begin{aligned}
&= \text{Im}_f(\{-1,1\}) \\
&= \{1\}
\end{aligned}
$$

So $\text{Im}_f(\text{PreIm}_f(Y)) \subsetneq Y$.

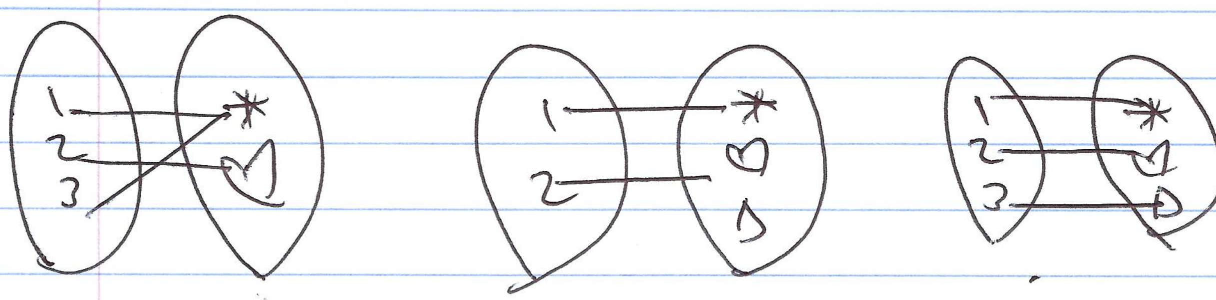
-jections

let

$$A = \{1, 2, 3\}$$
$$B = \{*, \heartsuit\}$$
$$C = \{1, 2\}$$
$$D = \{*, \heartsuit, \Delta\}$$

define $g : A \to B$
$h : C \to D$
$j : A \to D$

by: $g = \{(1,*), (2,\heartsuit), (3,*)\}$
$h = \{(1,*), (2,\heartsuit)\}$
$j = \{(1,*), (2,\heartsuit), (3,\Delta)\}$



g h j

Surjections

Def'n a function $f: A \to B$ is surjective (or onto) iff $Im_f = B$ that is, iff

$$(\forall b \in B)(\exists a \in A)\ f(a) = b$$

ex: - g, j above are surjective

- h is not: $\Delta \notin Im_h$

Proving surjectivity

ex: ① Define $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $f((m,n)) = m+n$

Claim f is surjective

pf: WTS $(\forall x \in \mathbb{Z})(\exists (m,n) \in \mathbb{Z} \times \mathbb{Z})$ s.t. $f(m,n) = x$

- So fix $x \in \mathbb{Z}$.
- observe $f(0,x) = 0 + x = x$
- hence $\exists (m,n)$ s.t. $f(m,n) = x$ namely $(m,n) = (0,x)$

- Since x was arbitrary, the theorem is proved.

(2) Define $f: \mathbb{R} \to \mathbb{R}$ by
$f(x) = 2x+1$

<u>Claim</u> f is surjective

<u>Pf:</u> - Fix $y \in \mathbb{R}$
- Let $x = \frac{y-1}{2}$

- then
$$f(x) = f\left(\frac{y-1}{2}\right)$$
$$= 2\left(\frac{y-1}{2}\right)+1$$
$$= y$$

- Since y was arbitrary, the claim is proved

(3) Define $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by
$f(m,n) = m+n$

<u>Claim:</u> f is <u>not</u> surjective

Pf: WTS:

$$(\exists x \in \mathbb{N})(\forall (m,n) \in \mathbb{N}\times\mathbb{N})\ f(m,n) \neq x$$

- Consider $x = 1$
- Fix $(m,n) \in \mathbb{N}$
- then since $m, n \geq 1$ we have $m+n \geq 2$
- that is, $f(m,n) \geq 2$.
- hence $f(m,n) \neq 1$
- Since (m,n) was arbitrary the claim is proved.

Injections

Def'n Suppose $f: A \to B$ is a function.
Then f is called injective (or 1-1, or one-to-one) iff
$$(\forall x, y \in A)(f(x) = f(y) \Rightarrow x = y)$$

↳ sometimes helpful to write this def'n in contrapositive form

$$(\forall x, y \in A)(x \neq y \Rightarrow f(x) \neq f(y))$$

"distinct inputs map to distinct outputs"

ex's – g above is not injective:
$1 \neq 3$ but $g(1) = g(3) = *$
– h, j are injective.

Proving injectivity

~~ex ① Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = 5x+6$~~

~~Claim f is injective~~

Two approaches:
Fix $x, y \in A$ and either
① Assume $f(x) = f(y)$, prove $x = y$
or ② Assume $x \neq y$, prove $f(x) \neq f(y)$

ex's ① Define $f: \mathbb{R} \to \mathbb{R}$ by
$f(x) = 5x + 6$.

Claim f is injective

Pf. – Fix $x, y \in \mathbb{R}$.
– assume $f(x) = f(y)$
– then $5x + 6 = 5y + 6$
$\Rightarrow 5x = 5y$
$\Rightarrow x = y$

- since x, y were arbitrary, claim is proved

(2) Define $f: \mathbb{N} \to \mathbb{N}$ by $f(n) = n^2$

Claim f is injective

Pf: - Fix ~~[illegible]~~ $n, m \in \mathbb{N}$

- Assume $n \neq m$. (wts: $f(n) \neq f(m)$)
- then either $n < m$ or $m < n$
- assume, without loss of generality, that $n < m$
- then since $n, m \in \mathbb{N}$ are both positive we may square both sides: to obtain $n^2 < m^2$
- i.e. $f(n) < f(m)$
- hence $f(n) \neq f(m)$ ✓
- claim is proved.

(3) Define: $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $f(m, n) = m + n$

Claim f is not injective

Pf: – Observe $f(1,3) = 4 = f(2,2)$
– but $(1,3) \neq (2,2)$
– hence f is not injective

Bijections

Def'n a function $f: A \to B$ is called bijective iff f is both injective and surjective.

ex: – g above is not a bijection (is a surjection, but not injection)
– h isn't either (is an injection, but not a surjection)
– j is a bijection

Proving bijectivity

ex ① Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = 3x - 1$

Claim f is a bijection
Pf Subclaim: f is an injection
Pf: Fix $x, y \in \mathbb{R}$ and assume
$f(x) = f(y)$

then $3x-1 = 3y-1$

$\Rightarrow 3x = 3y$

$\Rightarrow x = y$ ✓

subclaim proved ✓

<u>Subclaim</u>: f is a <u>surjection</u>

<u>Pf</u>: - fix $y \in \mathbb{R}$

- Let $x = \frac{y+1}{3}$

Then:

$$f(x) = f\left(\frac{y+1}{3}\right)$$

$$= 3\left(\frac{y+1}{3}\right) - 1$$

$$= y$$

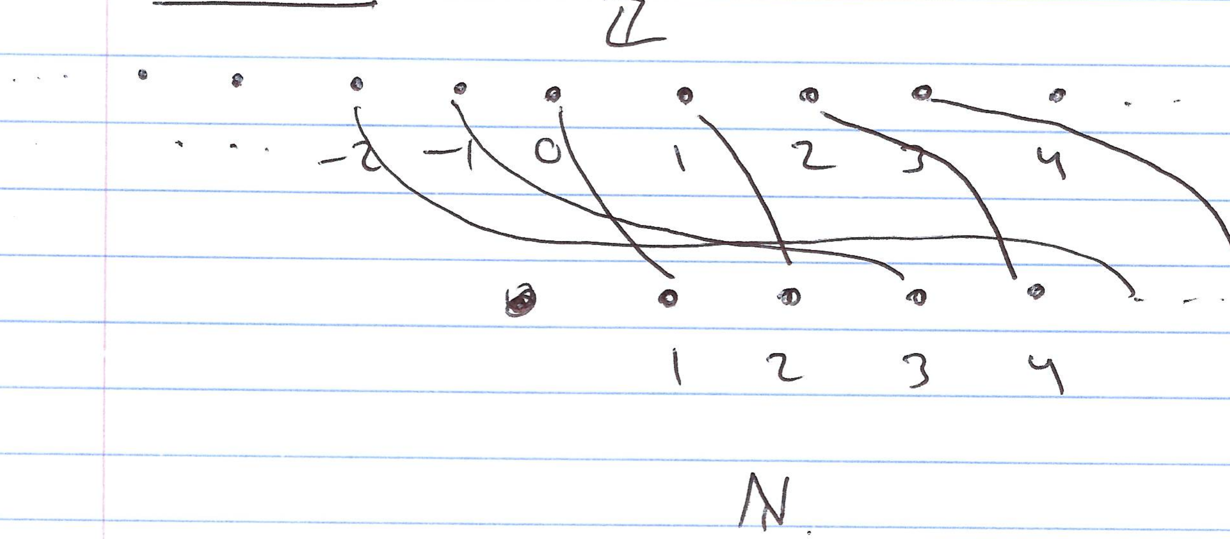
- since $y \in \mathbb{R}$ was arbitrary subclaim is proved ✓

Hence f is a bijection, as claimed ✓

(2) Define a function $f: \mathbb{Z} \to \mathbb{N}$ by:

$$f(n) = \begin{cases} 2n & \text{if } n > 0 \\ 2(-n)+1 & \text{if } n \leq 0 \end{cases}$$

Picture



Claim f is a bijection

Pf: surjectivity
- Fix $n \in \mathbb{N}$
- if n is even, then $n = 2k$ for some $k \in \mathbb{N}$ (so $k > 0$)
- hence $f(k) = 2k = n$

- if n is odd then $n = 2k+1$ for some $k \in \mathbb{N} \cup \{0\}$
- hence $-k \leq 0$
- hence $f(-k) = 2k+1 = n$

- hence in either case $\exists x \in \mathbb{Z}$ i.e. $f(x) = n$
- hence f is surjective ✓

(ii) <u>injectivity</u>

- Fix $n, m \in \mathbb{Z}$
  and suppose $n \neq m$
- WLOG assume $n < m$

<u>Case 1</u>: $0 < n < m$
- then $f(n) = 2n$ and $f(m) = 2m$
- since $n < m$, $2n < 2m$
- i.e. $f(n) < f(m)$
- hence $f(n) \neq f(m)$

<u>Case 2</u>: $n < m \leq 0$
- then $f(n) = 2(-n) + 1$
  $f(m) = 2(-m) + 1$
- since $n < m$ we have

$$-n > -m$$
$$\Rightarrow 2(-n) > 2(-m)$$
$$\Rightarrow 2(-n) + 1 > 2(-m) + 1$$

i.e. $f(n) > f(m)$
hence $f(n) \neq f(m)$

<u>Case 3</u>: $n \leq 0 < m$
then $f(n)$ is odd
and $f(m)$ is even
- hence $f(n) \neq f(m)$

- Thus in all 3 cases

$$f(n) \neq f(m)$$

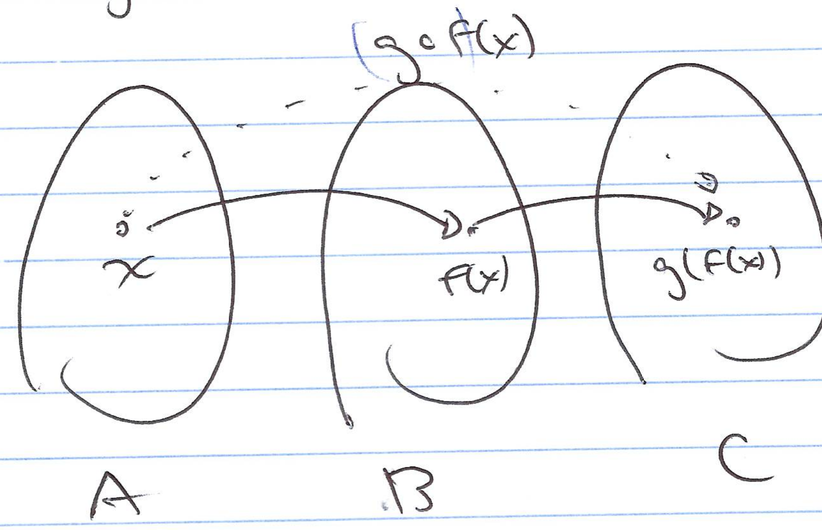
- hence f is injective ✓

- hence f is bijective ✓

Compositions + Inverses

Def'n Suppose $f: A \to B$ and $g: B \to C$ are functions. The composition of f and g, denoted $g \circ f$, is the function defined by, $\forall x \in A$,

$$g \circ f(x) = g(f(x))$$



ex - Define $f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ by
$f(m,n) = m+n$
- Define $g: \mathbb{Z} \to \mathbb{N}$ by
$g(n) = n^2+1$

Then: $g\circ f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{N}$

$$
\begin{aligned}
- g\circ f(2,1) &= g(f(2,1)) \\
&= g(3) \\
&= 3^2+1 = 10
\end{aligned}
$$

- In general

$$
\begin{aligned}
g\circ f(m,n) &= g(f(m,n)) \\
&= g(m+n) \\
&= (m+n)^2+1
\end{aligned}
$$

The identity

Def'n Let A be a set. The identity function on A, denoted id_A, is the (unique) function on A s.t. $\forall x \in A$

$$id_A(x) = x$$ "$f(x) = x$"

Note: $id_A : A \to A$.

<u>Inverses</u> <u>Def'n.</u> - Let $f: A \to B$ be a function. Then f is <u>invertible</u> iff there exists a $g: B \to A$ s.t.

$$g \circ f = id_A$$

and

$$f \circ g = id_B$$

- g is called the <u>inverse</u> of f and written f^{-1}.

Note: not all functions are invertible! In fact we have:

<u>Theorem</u> Let $f: A \to B$ be a function. Then f is invertible if and only if f is a bijection

<u>Pf</u>: ($\Rightarrow$) Suppose f is invertible. Let g be f's inverse. We prove f is a bijection

<u>surjectivity</u>: fix $y \in B$. Consider $x = g(y)$. Then

$$\begin{aligned} f(x) &= f(g(y)) \\ &= f \circ g(y) \\ &= id_B(y) = y \end{aligned}$$

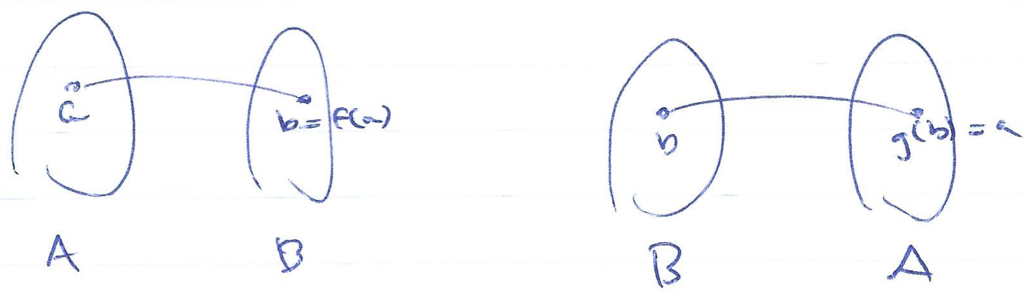
So $\exists\, x$ s.t. $f(x) = y$ (namely $y = g(x)$)

<u>injectivity</u> - Suppose $x, y \in A$ and $f(x) = f(y)$.

- then $g(f(x)) = g(f(y))$
- but since $g \circ f = id$ this means $x = y$ ✓

($\Leftarrow$) - Suppose f is a bijection from A to B.

- Define $g = \{(b,a) \in B \times A \mid (a,b) \in f\}$



We prove $g = f^{-1}$.

<u>Claim 1</u>: g is a function from B to A.

<u>pf</u>: - WTS $\forall b \in B$ there is a unique $a \in A$ s.t. $(b,a) \in g$

<u>existence</u> - Fix $b \in B$. Since f is <u>surjective</u>, $\exists a \in A$ s.t. $(a,b) \in f$. Hence $(b,a) \in g$.

<u>uniqueness</u> - Suppose $a' \in A$ s.t. $(b,a') \in g$. Then $(a',b) \in f$, i.e. $f(a') = b = f(a)$

Since f is <u>injective</u> must be $a = a'$ ✓

<u>Claim 2</u> $g = f^{-1}$.

<u>pf</u>: - Fix $a \in A$. Let $b = f(a)$

then $(a,b) \in f$

hence $(b,a) \in g$ i.e. $g(b) = a$

then $g(f(a)) = a$ so $g \circ f = id_A$

- Fix $b \in B$. Let $a = g(b)$
- So $(b,a) \in g$. Then $(a,b) \in f$

i.e. $f(a) = b$.

- So $f(g(b)) = b$
- hence $f \circ g = id_B$ ✓

Can use theorem to <u>prove</u> certain functions are bijections.

<u>Ex</u>: ① Define $f: \mathbb{R} \to \mathbb{R}$

by $f(x) = 2x - 5$

<u>Claim</u> f is a bijection

<u>pf</u>: - we'll show f has an inverse

- Let $g(x) = \frac{x+5}{2}$

Fix $x \in \mathbb{R}$

then $f \circ g(x) = f(g(x)) = 2\left(\frac{x+5}{2}\right) - 5$

$= x$

(c) $g \circ f(x) = g(f(x)) = \frac{(2x-5)+5}{2}$
$= x$

Hence $g \circ f = f \circ g = id_{\mathbb{R}}$
hence f is invertible
hence f is a bijection