Homework #9

1. Define functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ such that $g \circ f = \mathrm{id}_{\mathbb{N}}$ but $f \circ g \neq \mathrm{id}_{\mathbb{N}}$. Prove that your functions satisfy these identities.

2. Suppose that A and B are sets. Suppose further that $|A| = |B|$, that is, there exists a bijection $f : A \to B$. Show that $|\mathcal{P}(A)| = |\mathcal{P}(B)|$ by constructing a bijection $F : \mathcal{P}(A) \to \mathcal{P}(B)$. Prove that the function F you construct is a bijection.

3. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by the rule $f(x, y) = (3x + 2y, 4x + y)$. Show that f is a bijection by explicitly defining an inverse function for f and proving it is the inverse.

4. (Representing subsets of $\mathbb{N}$ as infinite strings).

   a. Prove that the set $A = \{f \subseteq \mathbb{N}^2 \mid f : \mathbb{N} \to \mathbb{N} \text{ is a function}\}$ is uncountably infinite by constructing an injection $F : \mathcal{P}(\mathbb{N}) \to A$. Prove that the function F you construct is an injection.

   b. Prove that the set $B = \{f \subseteq \mathbb{N} \times \{0, 1\} \mid f : \mathbb{N} \to \{0, 1\} \text{ is a function}\}$ is uncountably infinite by constructing a bijection $G : \mathcal{P}(\mathbb{N}) \to B$. Prove that the function G you construct is a bijection.

5. Let $C = \{f \subseteq \mathbb{R}^2 \mid f : \mathbb{R} \to \mathbb{R} \text{ is a function}\}$. Prove that $|\mathbb{R}| < |C|$ by showing there is no surjection $F : \mathbb{R} \to C$.