Homework #6

1. Let $P(n)$ be a variable proposition. In each of the following cases, assume that both BC (the base case(s)) and IP (the inductive principle) hold. Determine the largest subset $S \subseteq \mathbb{Z}$ for which, from these assumptions, we can conclude $(\forall n \in S)P(n)$.

   a. BC: $P(-3)$. IP: $(\forall n \in \mathbb{Z})(P(n) \Rightarrow P(n+1))$.

   b. BC: $P(1)$. IP: $(\forall n \in \mathbb{N})(P(n) \Rightarrow P(2n))$.

   c. BC: $P(0)$. IP: $(\forall n \in \mathbb{Z})(P(n) \Rightarrow P(n-1) \wedge P(n+1))$.

   d. BC: $P(0) \wedge P(1)$. IP: $(\forall n \in \mathbb{Z})(P(n) \Rightarrow P(n+3))$.

2. Prove that, for every $n \in \mathbb{N}$, the integer

   $$2 \cdot 7^n + 3 \cdot 5^n - 5$$

   is a multiple of 24.

3. Define a sequence a_n recursively, as follows:

   $$a_0 = 4, a_1 = 9, \text{ and } a_n = 5a_{n-1} - 6a_{n-2} \text{ for all } n \geq 2.$$

   Use strong induction to prove that, for all $n \in \mathbb{N} \cup \{0\}$, we have $a_n = 3 \cdot 2^n + 3^n$.

4. Let R be a relation defined on $\mathcal{P}(\mathbb{Z})$ defined by

   $$(A, B) \in R \text{ if and only if } A \cap B \neq \emptyset.$$

   Prove or disprove each of the following statements:

   a. R is reflexive.

   b. R is symmetric.

   c. R is transitive.

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function on $\mathbb{R}$. Define a relation R_f on $\mathbb{R}$ by the rule $(x, y) \in R_f$ if and only if $f(x) = f(y)$.

   a. Prove that R_f is an equivalence relation.

   b. Suppose that f is the squaring function defined by $f(x) = x^2$. For a fixed real number $r \in \mathbb{R}$, determine the equivalence class $[r]_{R_f}$.