Selection w/ repetition

Q: How many ways are there to select \( n \) objects from \( k \) types of objects, if repetition allowed.

Ex's: Dee's Donuts sells 4 types of donuts; you want to buy a dozen. How many distinct ways to do this?

Sol'n Imagine putting down 3 "spacers" to separate donut types.

\[
\begin{array}{cccccccc}
\quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\& \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\& \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\& \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\& \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\& \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\end{array}
\]

type 1 2 3 4

This "donut + spacer" diagram would correspond to purchase of:

- 3 donuts of type 1
- 2 " " " 2
- 5 " " " 3
- 2 " " " 4

→ Can view diagram as a 01-sequence w/ 12 0's (for donuts) 3 1's (to separate 4 types)
Conversely: any such sequence \((1203,315)\) corresponds to a selection of 12 donuts:

- \(0 + p1\) donuts
- \(1 + p2\) "
- \(7 + p3\) "
- \(4 + p4\) "

E.g.

\[
10100000000000000
\]

Corresponds to:

- \(0 + p1\) donuts
- \(1 + p2\) "
- \(7 + p3\) "
- \(4 + p4\) "

Hence: # of ways to make a selection = # of 01-seqs w/ 12 0's and 3 1's = # of 01-seqs of length 15 w/ 3 1's = \((15/3) = 455\)

Same reasoning in general proves:

**Theorem:** The # of ways to select \(n\) objects from \(k\) types w/ repetition allowed is:

\[
\binom{n + (k-1)}{k-1}
\]

\((k-1)\) because we need \(k-1\) "spacers" to separate \(k\) types of objects
Ex: Suppose we roll $n$ indistinguishable 6-sided dice.

How many distinct outcomes are possible?

Sol'n: Each of the $n$ dice can roll into 6 possible "types"

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
00 & 0 & 0 & 000 & 0 & 00
\end{array}
\]

Hence # of possible outcomes is:

\[
\binom{n+(6-1)}{6-1} = \binom{n+5}{5}
\]

So if we roll 10 dice, # is:

\[
\binom{15}{5} = 3003
\]

Countin' in Two Ways:

Then (Pascal's identity): Fix $n, k \in \mathbb{N}$

with $k \leq n$. Then:

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]

PF: Let $S$ be the set of $k$ element subsets of $\{1, 2, \ldots, n\}$.
Then \( |S| = (\binom{n}{k}) \)

**Observation:** We can partition \( S \) into \( S_i \) and \( T \) where:

- \( S_i = k \) element subsets of \( \{n\} \) that contain 1.
- \( T = k \) element subsets of \( \{n\} \) that don't contain 1.

Then: \( |S| = |S_i| + |T| \)

**Observe:** Subsets in \( S_i \) are formed by selecting \( k-1 \) elements from \( \{2, 3, \ldots, n\} \)

(1 is auto included) \( \Rightarrow |S_i| = \binom{n-1}{k-1} \)

Subsets in \( T \) are formed by selecting \( k \) elements from \( \{2, 3, \ldots, n\} \)

\( \Rightarrow |T| = \binom{n-1}{k} \)

**Hence:** \( |S| = \binom{n-1}{k-1} + \binom{n-1}{k} \)

so \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)
Claim: Fix $n, k \in \mathbb{N}$, $k \leq n$. Then:

$${n \choose k} k = n \frac{n-1}{k-1}$$

**Proof (PF):** Let $S$ denote set of committees of $k$ ppl chosen from a group of $n$ ppl, w/ a specified charperson:

- picking the committee members $\binom{n}{k}$
- from these, picking the chair $\binom{k}{1}$

\[ |S| = \binom{n}{k} \]  

or by:

- picking a chair first: $\binom{n}{1} = n$
- from remaining $n-1$ ppl, choose remaining $k-1$ committee members $\binom{n-1}{k-1}$

**Hence:**  

$$|S| = n \frac{n-1}{k-1} \quad \text{too}$$

\[ n \frac{n-1}{k-1} = \binom{n}{k} k \]
in this case can verify identity algebraically:

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}
\]

and \(n\binom{n-1}{k-1} = n \frac{(n-1)!}{(k-1)![n-1-(k-1)]!}
\]

\[
= \frac{n!}{(k-1)!(n-k)!}
\]

Same.

Prop'n: Fix \(n \in \mathbb{N}\). Then:

\[
n2^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k
\]

PF: Let \(S\) be the set of nonempty committees with a chairperson chosen from a group of \(n\) people.

Done. Why:
Can form a committee by:
- choosing the chair \((^n)_1 = n\)
- from remaining \(n-1\) ppl, choosing other committee members (i.e. just choose a subset from a set of size \(n-1\), \(2^{n-1}\))

\[
\Rightarrow |S| = n \cdot 2^{n-1}
\]

Alternatively, we can partition \(S\):

\[
S = A_1 \cup A_2 \cup \ldots \cup A_n
\]

where \(A_k\) is set of committees with exactly \(k\) ppl.

Before we computed:

\[
|A_k| = (^n)_k
\]

So

\[
|S| = |A_1| + |A_2| + \ldots + |A_n|
\]

\[
= (\binom{n}{1}) + (\binom{n}{2}) \cdot 2 + \ldots + (\binom{n}{n}) \cdot n
\]

\[
= \sum_{k=1}^{n} (\binom{n}{k}) \cdot k
\]