In general: \( g \circ f(mn) = g(f(mn)) \)
\[ = g(m+n) \]
\[ = (m+n)^2 + 1 \]

Observe: \( g \circ f: \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{N} \)

The identity function

\[ \text{Defn: } \text{Sp: } A \ni a \rightarrow a. \text{ The identity function on } A, \text{ denoted } \text{id}_A, \text{ is defined by:} \]
\[ \text{id}_A: A \rightarrow A \]
\[ \text{id}_A(x) = x \]

E.g. if \( A = \{*, 0, A\} \) then \( \text{id}_A: A \rightarrow A \)
\[ \text{is: } \]
\[ \{(*, *), (0, 0), (A, A)\} \]
Defn a function \( f: A \to B \) is called invertible if there is a function \( g: B \to A \) s.t.:

\[
g \circ f = \text{id}_A \quad \text{i.e.} \quad (\forall x \in A) \quad g(f(x)) = x
\]

and

\[
f \circ g = \text{id}_A \quad \text{i.e.} \quad (\forall y \in B) \quad f(g(y)) = y
\]

If such \( g \) exists it is called the inverse of \( f \) and denoted \( f^{-1} \)

**Ex:** Consider \( f: \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = 2x + 1 \)

Let \( g: \mathbb{R} \to \mathbb{R} \) be defined by:

\[
g(x) = \frac{x - 1}{2}
\]
Observe: \((\forall x \in \mathbb{R}) \ g(f(x)) = g(2x+1)\)
\[= \frac{2x+1 - 1}{2} = x\]

hence \(g \circ f = \text{id}_{\mathbb{R}}\)

also: \((\forall x \in \mathbb{R}) \ f(g(x)) = f\left(\frac{x-1}{2}\right)\)
\[= 2 \left(\frac{x-1}{2}\right) + 1 = x\]

hence \(f \circ g = \text{id}_{\mathbb{R}}\). Thus, \(f\) is invertible and it's inverse \(f^{-1}\) is \(g\). Note: not all functions are invertible.

In fact:

Theorem: a function \(f: A \rightarrow B\) is invertible iff \(f\) is a bijection.

Pf: \((\Rightarrow)\) Suppose \(f\) is invertible and let \(g = f^{-1}\) be its inverse.

\(\therefore f\) is a bijection.
(surjectivity) - fix \( y \in B \)
- let \( x = g(y) \)
- then \( f(x) = f(g(y)) = y \)
  (since \( f \circ g = \text{id}_B \))

Since \( y \in B \) was arbitrary, \( f \) is surjective.

(injectivity) - fix \( x, y \in A \) and \( x \neq y \)
- then \( g(f(x)) = g(f(y)) \)
- hence \( x = y \) (since \( g \circ f = \text{id}_A \))

Since \( x, y \in A \) were arbitrary, \( f \) is injective.

(\( \Rightarrow \)) \( \Rightarrow \) since now that \( f: A \to B \) is a bijection

WTS: \( f \) is invertible.

Define: \( g = \{ (b, a) \in B \times A \mid (a, b) \in f \} \)

Claim: \( \circ \) \( g \) is a function from \( B \to A \)
\( \circ \) \( g = f^{-1} \)
Proof (1): \text{WTS: } (\forall b \in B) \exists a \text{ unique } a \in A

5.1. \quad (b, a) \in g.

- so fix \, b \in B

(Existence) - since \, f \, is surjective, \forall a \in A

\quad 5.1. \quad f(a) = b, \, \text{i.e. } (a, b) \in F

- hence \, (b, a) \in g, \, \text{by definition of } g

(Uniqueness) - so there is only one \, a \in A \, 5.1.

(b, a') \in g \, \text{as well.}

- then \, (a', b) \in F

- i.e. \quad f(a') = b

- but then \quad f(a') = f(a). \, \text{Since } f

is injective, must have \, a' = a

Hence \, \text{\circ \, is proved} \checkmark

Proof (2): \text{WTS: } g = f^{-1} \quad \text{i.e. } g \circ f = \text{id}_A

f \circ g = \text{id}_B

- so fix \, a \in A

- let \, b = f(a) \, \text{so that } (a, b) \in F

- then \, (b, a) \in g \, \text{i.e. } g(b) = a

- i.e. \quad g(f(a)) = a

- since \, a \in A \, \text{was arbitrary, } g \circ f = \text{id}_A
- Now fix $b \in B$. Let $a = g(b)$ so that $(b, a) \in g$
- then must be $(a, b) \in f$, i.e. $f(a) = b$
- hence $f(g(b)) = f(a) = b$
- since $b \in B$ was arbitrary, $fog = 1_B$. Hence $g = f^{-1}$ as claimed.
- Hence $f$ is invertible.

We can sometimes use theorem to prove a given $f$ is a bijection.

ex: Consider $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$

Claim: $f$ is a bijection

Pf: we checked above that $g(x) = \frac{x-1}{2}$ then $g = f^{-1}$

Hence $f$ is invertible

By theorem, $f$ is a bijection