Infinity

The concept of cardinality:
- We would say that the set \{\ast, 0, 0\} has 3 elements or is of size 3.
- Why? By counting it!

\{\ast, 0, 0\}

1 2 3

- In doing so we are implicitly defining a bijection between \{1, 2, 3\} and \{\ast, 0, 0\}.

- We could've counted differently:

\{\ast, 0, 0\}

2 3 1

- Generalizing this observation:
we'll say two sets have the same size if there is a bijection between them.
Two sets $A, B$ have the same cardinality if there is a bijection $f: A \rightarrow B$. In this case we write $A \sim B$.

Note: In set theory courses, one defines, for every set $A$, the cardinal number $|A|$ can then prove: $A \sim B$, if $|A| = |B|$ (e.g. $|\ast, 0, \emptyset| = |\{1, 2, 3\}| = 3$)
- defining cardinal #s beyond our scope:
- for us $|A| = |B|$ just means $A \sim B$, i.e. $\exists f: A \rightarrow B$ bijective

Properties of $\sim$:
1. For any set $A$,
   - $\text{id}_A: A \rightarrow A$ is a bijection (Why?)
   - Hence $A \sim A$, i.e. $\sim$ is reflexive
2. If $f: A \rightarrow B$ is a bijection then $f$ is invertible and $f^{-1}: B \rightarrow A$ is a bijection too (Why?)
   - Hence if $A \sim B$ then $B \sim A$
   - i.e. $\sim$ is symmetric
3. On how you proved:
   If \( f: A \to B \) are bijections
   \( g: B \to C \)
   then \( gof: A \to C \) is a bijection too.

   - Hence if \( \sim \) and \( \sim \) then \( \sim \)
   - i.e. \( \sim \) is transitive

\( \Rightarrow 1 + 2 + 3 \) show \( \sim \) is an equiv.
relation on sets!

\( \sim \) most interesting when the sets
being compared are infinite

Ex's

1. \( \{1,2,3\} \sim \{\ast,0,\Delta\} \) since \( f = \{(1,\ast), (2,0), (3,\Delta)\} \) is a bijection.

2. Let \( \sim \) denote the set \( \{-1,-2,\ldots\} \)
Define \( F: N \to N \) by \( F(n) = -n \)
- Easy to check: \( F \) is a bijection
- Hence \( N \sim N \)

3. We showed before: \( f: \mathbb{Z} \to N \) defined by:
   \[ f(n) = \begin{cases} 
   2n & \text{if } n > 0 \\
   2(-n+1) & \text{if } n \leq 0 
   \end{cases} \]
   is a bijection. Hence \( \mathbb{Z} \sim N \)
4. Combining 2 and 3 gives $\sim \sim \sim \sim$, by transitivity of $\sim$.

Defn: Let $A$, $B$ be sets.

1. We write $A \leq B$ (or $|A| \leq |B|$) to mean: there is an injection $f: A \rightarrow B$.

2. We write $A \geq B$ (or $|A| \geq |B|$) to mean: there is a surjection $g: A \rightarrow B$.

3. We write $A < B$ to mean: there is an injection $f: A \rightarrow B$ but no bijection (i.e., $A \leq B$ but $A \not\sim B$).

4. Similarly $A > B$ means there is a surjection $g: A \rightarrow B$ but no bijection (i.e., $A \sim B$ but $A \not\sim B$).

Note: $A \geq B$ is not simply the "reverse of" $A \leq B$, i.e., it is not asserting there is an injection from $B$ to $A$. But this follows!

Theorem: For all sets $A$, $B$ we have:

$A \leq B$ iff $B \geq A$.

i.e., there is an injection $f: A \rightarrow B$ iff there is a surjection $g: B \rightarrow A$.
Before proving theorem, let's illustrate the ideas of proof w/ examples

Ex: 1 Consider \( f : \{2\} \to \{5\} \)

defined by \( f(1) = 1 \quad f(2) = 2 \quad f(3) = 3 \)

Observe: \( f \) is an injection, hence \([2] \subseteq [5]\)

Idea: to show \([5] \not\subseteq [2]\), i.e. to get a surjection \( g : [5] \to [2] \) we can "reverse" \( f \), then map anything left over to something arbitrary.

E.g. Let \( g \) such that:
\[
\begin{align*}
g(1) &= 1 \\
g(2) &= 2 \\
g(3) &= 3 \\
g(4) &= 1 \\
g(5) &= 1
\end{align*}
\]

Then: \( g \) is a function since \( f \) was injective (why?) and is clearly surjective.

More generally can prove: if \( A \subseteq B \) then \( B = A \) using this idea.
2. Consider \( g : \{3\} \rightarrow \{5\} \) defined by:

\[
\begin{align*}
  g(1) &= g(2) = 1 \\
  g(3) &= g(4) = 2 \\
  g(5) &= 3
\end{align*}
\]

Observe: \( g \) is a surjection.

Hence \( \{3\} \geq \{5\} \).

Idea: to show \( \{3\} \leq \{5\} \), i.e. to get an injection \( f : \{3\} \rightarrow \{5\} \) we take some "reverse" of \( g \) w/o repeats.

E.g. define \( f : \{3\} \rightarrow \{5\} \) by:

\[
\begin{align*}
  f(1) &= 1 \\
  f(2) &= 3 \\
  f(3) &= 5
\end{align*}
\]

\( f \) "reverses" \( g \) w/o repeats.

Then: \( f \) is a function because \( g \) was surjective and we deleted repeats (why?).

And it's injective because \( g \) was a function (why?).

More generally, we can prove: if \( A \geq B \) then \( B \leq A \) using some idea.