Ex: Q: for which \( n \in \mathbb{N} \) do we have \( 2^n < n! \)? Let's see...

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 2^n )</th>
<th>( n! )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>24</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>120</td>
</tr>
</tbody>
</table>

Seems like: if \( n \geq 4 \) then \( n! > 2^n \) let's prove.

Prop'h For every \( n \in \{4, 5, 6, \ldots\} \) we have \( n! > 2^n \)
(here: \( n_0 = 4 \), \( S = \{4, 5, 6, \ldots\} \))

PF: Let \( P(n) \) be the prop'h "\( n! > 2^n \)"

(BC) \( P(4) \) holds since \( 4! > 2^4 \)

(IH) Fix \( n \in \mathbb{N}, n \geq 4 \) and assume \( P(n) \) holds, i.e., assume \( n! > 2^n \)

note: we fix \( n \geq 4 \) not \( n > 4 \).
(IS) Then:

\[(n+1)! = n! \cdot (n+1)\]

\[> 2^n \cdot (n+1) \quad \text{by IH}\]

\[> 2^n \cdot 2 \quad \text{since } n \geq 4\]

\[= 2^{n+1}\]

We've shown \((n+1)! > 2^{n+1}\), i.e. \(P(n+1)\)

By induction we've proved for every \(n \geq 4\) (i.e. for \(n \in \{4, 5, 6, \ldots\}\)) we have \(n! > 2^n\)

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Induction w/ Jumps:

- Sometimes we want to prove \(P(n)\) for all \(n\),

  but when \(n\) is even,
  or... when \(n\) is odd,
  or... when \(n\) is a multiple of \(7\),
  etc.

  Can still argue inductively.

Then let \(P(n)\) be a var. prop'n.

Fix \(n_0 \in \mathbb{Z}\) and \(k \in \mathbb{N}\)

\(\text{or } \text{"starting pF" } \text{"jump"}\)
\[ S = \{ n_0, n_0 + k, n_0 + 2k, \ldots \} \]
\[ \text{If we have } 1 \ P(n_0) \]
\[ \text{and } 2 \ \forall n \in S \ (P(n) \Rightarrow P(n + k)) \]
\[ \Rightarrow \ (\forall n \in S \ P(n)) \text{ holds.} \]

Eg. \[ S = \{ 2, 4, 6, \ldots \} \]
and we can show
\[ 1 \ P(2) \]
\[ 2 \ \text{If } P(n), \text{ then } P(n + 2) \]
Then we've proved \( P(n) \) holds \( \forall n \in S \).

Ex. Consider the alternating sum of the first \( n \) squares:
\[ 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} n^2 \]
\[ = \sum_{k=1}^{n} (-1)^{k-1} k^2 \]

Proof.
1. If \( n \) is odd we have:
\[ \sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k \]
\[ (= \frac{n(n+1)}{2} \text{ by before}) \]

2. If \( n \) is even we have:
\[ \sum_{k=1}^{n} (-1)^{k-1} k^2 = -\sum_{k=1}^{n} k \]
\[ (= -\frac{n(n+1)}{2}) \]
PF: If here \( n=1 \) and jump = 2
so that \( S = \{1, 3, 5, \ldots \} \).

\[
P(n) \text{ is } \sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k
\]

(BC) If \( n=1 \),
\[
\sum_{k=1}^{1} (-1)^{k-1} k^2 = 1^2 = 1 = \sum_{k=1}^{1} k
\]

So \( P(n) \) holds.

(IH) Fix \( n \in \{1, 3, 5, \ldots \} \) and assume
\[
P(n), \text{ i.e. } \sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k
\]

(IS) Now consider the \( n+2 \) sum:
\[
\sum_{k=1}^{n+2} (-1)^{k-1} k^2 = \sum_{k=1}^{n} (-1)^{k-1} k^2 + (-1)^n (n+1)^2
\]
\[
= \sum_{k=1}^{n} (-1)^{k-1} k^2 - (n+1)^2 + (n+2)^2
\]
\[
= \sum_{k=1}^{n} k - (n+1)^2 + (n+2)^2 \quad \text{odd}
\]
\[
= \sum_{k=1}^{n} k + [ (n+2) - (n+1) ] [ (n+2) + (n+1) ]
\]
\[
= \sum_{k=1}^{n} k + (n+1) + (n+2) = \sum_{k=1}^{n} k
\]
So $P(n+2)$ holds. By induction we've proved, $\forall n \in \{1, 3, 5, \ldots\}$
\[
\sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k
\]

Summary: we showed

1. $P(1)$ holds
2. If $n \in \{1, 3, 5, \ldots\}$ then $P(n) \Rightarrow P(n+2)$

It follows: $P(n)$ holds $\forall n \in \{1, 3, 5, \ldots\}$.

2. For $n$ even: a similar arg. You try!
Outline is: 1. verify $P(2)$, i.e.
\[
\sum_{k=1}^{2} (-1)^{k-1} k^2 = \sum_{k=1}^{2} k
\]
2. Fix $n \in \{2, 4, 6, \ldots\}$, and assume $\sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k$
3. Use this to prove
\[
\sum_{k=1}^{n+2} (-1)^{k-1} k^2 = -\sum_{k=1}^{n} k
\]
Fibonacci sequence is defined recursively by:

\[ f_0 = 0 \quad f_1 = 1 \]
\[ f_n = f_{n-2} + f_{n-1} \quad \text{for } n > 2 \]

0, 1, 1, 2, 3, 5, 8, 13, 21, ...

\[ f_0 \quad f_1 \quad f_2 \quad f_3 \quad f_4 \quad \ldots \]

\[
\text{Fib. sequence is a playground for inductive proofs.}
\]

Prop’n \( \text{Then } n \in \mathbb{N} \), we have:

\[
\sum_{k=1}^{n} f_k = f_{n+2} - 1
\]

(i.e. \( f_1 + f_2 + \ldots + f_n = f_{n+2} - 1 \))

Pf: (BC) if \( n = 1 \) we have:

\[
\sum_{k=1}^{1} f_k = f_1 = 1 = 2 - 1 = f_3 - 1
\]

(IIt) Fix \( n \in \mathbb{N} \) and assume

\[
\sum_{k=1}^{n} f_k = f_{n+2} - 1
\]