PMF says: if you can do \(0, 1, 2\) then \((\forall n \in \mathbb{N}) P(n)\) holds.

**Ex:** \(Q:\) what is the sum of the first \(n\) natural numbers?

\[1 + 2 + \ldots + n = ?\]

**First few:**

\[
\begin{align*}
1 & = 1 = 1 \cdot \frac{2}{2} \\
1 + 2 & = 3 = 2 \cdot \frac{3}{2} \\
1 + 2 + 3 & = 6 = 3 \cdot \frac{4}{2} \\
1 + 2 + 3 + 4 & = 10 = 4 \cdot \frac{5}{2} \\
\end{align*}
\]

\[1 + 2 + \ldots + n = \frac{n(n+1)}{2},\]

**Theorem** For every \(n \in \mathbb{N}\), we have:

\[
\sum_{k=1}^{n} k = \frac{n(n+1)}{2}
\]

**PF:** Let \(P(n)\) be the property

\[
\sum_{k=1}^{n} k = \frac{n(n+1)}{2}
\]

**(BC):** \(P(1)\) is true since

\[
\sum_{k=1}^{1} k = 1 = \frac{1 \cdot 2}{2} \checkmark
\]
(III) Fix \( n \in \mathbb{N} \) and assume \( P(n) \), i.e. assume \( \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \).

(III) Now consider:

\[
\sum_{k=1}^{n+1} k = 1 + 2 + \ldots + n + n + 1
\]

\[
= \sum_{k=1}^{n} k + (n+1)
\]

by III

\[
= \frac{n(n+1)}{2} + (n+1)
\]

\[
= \frac{n(n+1) + 2(n+1)}{2}
\]

\[
= \frac{n(n+1) + 2(n+1)}{2}
\]

\[
= \frac{(n+1)(n+2)}{2}
\]

\[
= \frac{(n+1)(n+1+1)}{2}
\]

hence \( P(n+1) \) holds.

by PMI, \( P(n) \) holds for every \( n \in \mathbb{N} \).

i.e. \( \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \) \( \checkmark \)
Notice: proof doesn't really give insight into how we might have guessed the final:
\[ \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \]

But: once we have guessed, PMI gives us a way of verifying that it's really true for \( n \).

2. (Geometric series) Fix \( x \in \mathbb{R} \) with \( x \neq 0,1 \). Then for every \( n \in \mathbb{N} \) we have:

\[ 1 + x + x^2 + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1} \]

i.e.
\[ \sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1} \]

Call this \( P(n) \)

(RC) \( P(1) \) holds since
\[ \sum_{k=0}^{0} x^k = x^0 = 1 = \frac{x^1 - 1}{x - 1} \]

Since \( x \neq 1 \)

\[ \text{Since } x \neq 0 \]
(II) Fix \( n \in \mathbb{N} \) and assume \( P(n) \)

\[
\text{i.e., assume that } \sum_{k=0}^{\frac{n-1}{x^k}} = \frac{x^n - 1}{x-1}
\]

(II) Now consider:

\[
\sum_{k=0}^{n} x^k = \sum_{k=0}^{n-1} x^k + x^n
\]

by II,

\[
= \frac{x^n - 1}{x-1} + x^n
\]

\[
= \frac{x^n - 1 + x^{n+1} - x^n}{x-1}
\]

\[
= \frac{x^{n+1} - 1}{x-1}
\]

\[\Rightarrow P(n+1) \text{ holds}\]

By PMI, \( P(n) \) holds for all \( n \in \mathbb{N} \)

\[
\text{i.e., for all } n \in \mathbb{N} \text{ we have:}
\]

\[
\sum_{k=0}^{\frac{n-1}{x^k}} = \frac{x^n - 1}{x-1}
\]
Prop'n For every \( n \in \mathbb{N} \), \( 7^n - 4^n \) is a multiple of 3.

**Proof** (by mathematical induction)

1. **Base Case**: If \( n = 1 \), the statement holds since \( 7^1 - 4^1 = 3 \).

2. **Inductive Step**: Fix \( n \in \mathbb{N} \) and assume \( \exists k \in \mathbb{N} \) s.t. \( 7^n - 4^n = 3k \).

3. **Inductive Hypothesis (IH)**: Now, observe:

\[
7^n = 3k + 4^n \quad \text{(by IH)}
\]

\[
\Rightarrow 7^{n+1} = (3k + 4^n)7
\]

\[
= 21k + 7 \cdot 4^n
\]

\[
= 21k + (3+4) \cdot 4^n
\]

\[
= 21k + 3 \cdot 4^n + 4^{n+1}
\]

\[
\Rightarrow 7^{n+1} - 4^{n+1} = 21k + 3 \cdot 4^n = 3(7k+4^n)
\]

\[
= 3M
\]

where \( M = 7k+4^n \).

Hence, \( 7^{n+1} - 4^{n+1} \) is a multiple of 3.

By PMI, \( 7^n - 4^n \) is a multiple of 3 \( \forall n \in \mathbb{N} \).
Variants of induction

Nothing special about \( n=1 \) as a base case.

Thus (PMI with different BC)

- Sps \( P(n) \) is a var. prop'n and \( n \in \mathbb{Z} \), possibly 0 or negative

- \( \mathbb{W} S = \{ n_0, n_0 + 1, n_0 + 2, \ldots \} = \{ n \in \mathbb{Z} | n \geq n_0 \} \)

Then if we have

1. \( P(n_0) \) holds
2. \( \forall n \in S \) \( P(n) \Rightarrow P(n+1) \) holds

then \( \forall n \in S \) \( P(n) \) holds.

\( \Rightarrow \) can prove theorem using regular PMI (see book)

\( \Rightarrow \) proof template nearly the same as w/ PMI:

1. (BC) verify \( P(n_0) \)
2. (IH) Fix \( n \in S \). Assume \( P(n) \)
3. (IH) Prove \( P(n+1) \)