Then, the following are equivalent:

1. PMI
2. PSMI
3. WOP

i.e. \( PMI \iff PSMI \iff WOP \)

i.e. From any of these statements, we can prove the other two.

If: We've already shown:

\[ PSMI \Rightarrow WOP \]

Hence, if we can show:

Claim 1: \( PMI \Rightarrow PSMI \)

Claim 2: \( WOP \Rightarrow PMI \)

We will have established the equivalence of all three statements.

Before proving \( PMI \Rightarrow PSMI \), let's illustrate the idea of proof w/ an example.
Recall our first strong induction proof:

Define $S_0 = 1$

$S_n = 1 + \sum_{k=0}^{n-1} S_k$ for $n > 1$

Claim: Then we have $S_n = 2^n$

PF: We proved this with PSMI $P(n)$

- To prove using just PMI we can "hack" a strong inductive hypothesis into the statement we induct on.

- Let $Q(n)$ be "$(\forall k \in \{1, \ldots, n\})(S_k = 2^k)$".

- If we can prove $Q(n)$ holds for all $n$, then in particular we've proved $S_n = 2^n$ holds for all $n$, i.e. $P(n)$ holds for all $n$.

(BC) $Q(0)$ holds since this is just $(\forall k \in \{0\})(S_k = 2^k)$, which is true since $S_0 = 1 = 2^0$. 
(IH) Fix \( n \in \text{Nu}(0) \) and assume \( Q(n) \).

i.e. assume \((\forall k \in \{0, \ldots, n\}) (f_k = 2^k)\)

(IS) to prove \( Q(n+1) \), i.e.

\((\forall k \in \{0, \ldots, n, n+1\}) (f_k = 2^k)\)

It is sufficient to prove \( S_{n+1} = 2^{n+1} \).

Since our IH already gives \((\forall k \in \{0, \ldots, n\}) f_k = 2^k\).

Observe: \( S_{n+1} = 1 + \sum_{k=0}^{n} f_k \)

Define:

\[
S_n = 1 + \sum_{k=0}^{n} 2^k \quad \text{by IH}
\]

\[
= 1 + \frac{2^{n+1} - 1}{2-1}
\]

\[
= 2^{n+1}
\]

So by regular induction we've proved

\((\forall n \in \text{Nu}(0)) Q(n) \) which as noted gives

\((\forall n \in \text{Nu}(0)) p(n) \) i.e. \((\forall n \in \text{N}) S_n = 2^n \).
Proof of $PMI \Rightarrow PSMI$

Assume $PMI$: for every prop'n $P(n)$, if $\circ P(i)$ and $\circ \theta \circ (\forall n \in \mathbb{N}) (P(n) \Rightarrow P(n+1))$ hold, then $(\forall n \in \mathbb{N}) P(n)$ holds.

Want to prove: $PSMI$: for every prop'n $Q(n)$, if $\circ \theta \circ Q(i)$ and $\circ \theta \circ (\forall n \in \mathbb{N})[\circ \theta \circ (\forall k \in \mathbb{N}) Q(k) \Rightarrow Q(k+1)]$ hold, then $(\forall n \in \mathbb{N}) Q(n)$ holds.

- so fix a prop'n $Q(n)$ and assume $\circ$ and $\theta$ hold.
- let $P(n)$ be the prop'n
  
  $(\forall k \in \mathbb{N}) Q(k) \Rightarrow Q(k+1)$

We'll prove $(\forall n \in \mathbb{N}) P(n)$ by $PMI$.

$(P)$ $P(i)$ holds, since this is $\theta \circ (\forall k \in \mathbb{N}) Q(k)$, which is equiv. to $Q(i)$, which holds by assumption $\circ$.

$(IH)$ Fix $n \in \mathbb{N}$ and assume $P(n)$, i.e., assume $(\forall k \in \mathbb{N}) Q(k)$ holds.

$(IS)$ Then by $\theta$ we know $Q(n+1)$ holds; hence $(\forall k \in \mathbb{N}) Q(k)$ and $Q(n+1)$ hold, hence $(\forall k \in \mathbb{N}) Q(k)$ holds, i.e. $P(n+1)$ holds.
by (regular) induction (PMI) we have \((\forall n \in \mathbb{N}) \ P(n)\) holds, i.e. 
\((\forall n \in \mathbb{N}) \ (\forall k \in \mathbb{N}) \ Q(k)\) holds. But this implies \((\forall n \in \mathbb{N}) \ Q(n)\) holds, as desired.

Proof of \(\text{WOP} \Rightarrow \text{PMI}\): Assume \(\text{wop}\).

We want to prove \(\text{PMI}\). So fix a prop'n \(P(n)\) and assume

1. \(P(1)\)
2. \((\forall n \in \mathbb{N}) \ (P(n) \Rightarrow P(n+1))\) both hold.

We prove \((\forall n \in \mathbb{N}) \ P(n)\) holds.

(idea: let \(S = \{n \in \mathbb{N} \mid P(n)\} \text{ fail}\).

we'll use \(\text{wop}\) to prove \(S = \emptyset\).

- if \(S \neq \emptyset\), then by \(\text{wop}\), \(S\) has a least element \(x\).
- we know \(x \neq 1\), since \(P(1)\) holds.
- hence \(x = n + 1\) for some \(n \in \mathbb{N}\).
- since \(x\) is least number for which \(P(n)\) is false, must have that \(P(n)\) holds.
- but then by $\exists$, $P(n+1)$ holds, i.e. $P(x)$ holds, a contradiction, as $\neg P(x)$ fails.
- Hence $S = \emptyset$, which gives (the NIPR) holds, as desired.

QPMI, PSMI, WOP are all intuitively obvious principles, and often taken as axioms.

As we've proved: if you assume any one of them, can prove other two.
**Binary relations**

- Binary relations are ubiquitous in math.
- E.g., we have order relations like:
  \[ x \leq y \]
  \[ x < y \]

- The subset relation
  \[ X \subseteq Y \]

- The divisibility relation
  \[ n \mid m \] ("n divides m")

- All assert a relation between two objects (hence "binary")
- But what are \( <, \leq, \geq, = \) as math's objects themselves?

- We will define binary relations as sets of ordered pairs

  **Def** n Sps \( A, B \) are sets. A binary relation on \( A \) and \( B \) is simply a subset \( R \subseteq A \times B \).
If \((a, b) \in R\) we say “\(a\) is related to \(b\)” and sometimes write \(aRb\).

If we say \(A\) is the domain of \(R\), \(B\) the codomain of \(R\).

It often we have \(A = B\) so that \(R \subseteq A \times A\). In this case we say: \(R\) is a relation on \(A\).

Example 1: Let \(A = \) set of Shakespeare’s characters, \(B = \) set of Shakespeare’s plays.

- Define a relation \(R \subseteq A \times B\) by:
  \[(a, b) \in R \text{ iff } a \text{ appears in } b.\]

  I.e. \(R = \{(a, b) \in A \times B \mid a \text{ appears in } b\}\)

- Then: \((\text{Romeo}, \text{“Romeo and Juliet”}) \in R\)
  \((\text{Iago}, \text{“Othello”}) \in R\)
  \((\text{Romeo}, \text{“Othello”}) \notin R\).

- Might also write: \(\text{Romeo} R \text{“Romeo and Juliet”}\)
  \(\text{Romeo} R \text{“Othello”}\)

Example 2: Consider \(<\) and \(\leq\) as relations on \(\mathbb{N}\).

Can think of them as sets of pairs.