**Direct proof strategy:** define a specific yes and prove \( P(y) \) holds.

*Ex: 1-* Propin There exists an even integer \( n \in \mathbb{N} \) that can be written as the sum of two primes in two distinct ways.

*Pf:* Consider \( n = 10 \). Then \( n \) is even and moreover \( 10 = 5 + 5 = 7 + 3 \). Since \( 3, 5, 7 \) are prime, the claim is proved. (Note: \( 24 = 19 + 5 = 17 + 7 \) works too.)

**Indirect proof strategy:** Assume \( \neg(\exists x \in \mathbb{N}) P(x) \) (equiv. \( \forall x \in \mathbb{N} \neg P(x) \)) and derive a contradiction.

*Ex: 2-* Fix \( n \in \mathbb{N} \) and \( a_1, \ldots, a_n \in \mathbb{R} \). Then there is a \( k \in \{1, 2, \ldots, n\} \) s.t. \( a_k \) is at least as large as the average (mean) of \( a_1, \ldots, a_n \), i.e.: 
\[
(\exists k \in \{1, 2, \ldots, n\})(a_k \geq \frac{1}{n}(a_1 + a_2 + \ldots + a_k))
\]
Proof: -Suppose, toward a contradiction.
- That is, suppose that

\[(\forall k \in \mathbb{N})(a_k < \frac{1}{n}(a_1 + \ldots + a_n))\]

- For simplicity, let \( s = a_1 + a_2 + \ldots + a_n \)

- So our assumption is: \((\forall k \in \mathbb{N})(a_k < \frac{s}{n})\)

- But then:

\[s = a_1 + a_2 + \ldots + a_n \quad \text{(defn of } s)\]

\[< \frac{s}{n} + \frac{s}{n} + \ldots + \frac{s}{n}\]

\[= n \cdot \frac{s}{n} = s \quad \text{(by our assumption)}\]

This shows \( s < s \), a contradiction.
Hence our assumption was false, and so the proposition is true.

Universal Claims
General Form: \((\forall x \in S) P(x)\)
Direct strategy: - Let \( x \in \mathbb{R} \) be arbitrary but fixed. Prove \( P(x) \).

Ex: 0 Prop'n \( (\forall x, y \in \mathbb{R}) (xy \leq \left( \frac{x+y}{2} \right)^2) \)

Pf: - Fix \( x, y \in \mathbb{R} \).
- then: \( (x-y)^2 \geq 0 \) (square always \( \geq 0 \))
- i.e. \( x^2 - 2xy + y^2 \geq 0 \)
\[ \Rightarrow x^2 + y^2 \geq 2xy \]
\[ \Rightarrow x^2 + 2xy + y^2 \geq 4xy \]
- i.e. \( (x+y)^2 \geq 4xy \)
\[ \Rightarrow \frac{(x+y)^2}{4} \geq xy \]
\[ \Rightarrow \left( \frac{x+y}{2} \right)^2 \geq xy \], as desired.

Since \( x, y \in \mathbb{R} \) were arbitrary the claim is proved. \( \checkmark \)

Aside: prop'n is one version of the "AM - GM" inequality.
- arithmetic mean (AM) of \( xy \) is \( \frac{x+y}{2} \)
- geometric mean (GM) of \( xy \) is \( \sqrt{xy} \)
The problem says (for \( x, y > 0 \)) that
\[
\sqrt{xy} \leq \frac{x+y}{2},
\]
i.e. GM \( \leq \) AM.

**Indirect strategy**: Assume \( \neg (\exists x, s.t. P(x)) \)
(equiv: \( \neg (\exists x, \neg P(x)) \)) and derive a contradiction.

**Ex**: \( \sqrt{2} \) is irrational, then \( u: \)
\[
(\forall a, b \in \mathbb{Z})(\frac{a}{b} \neq \sqrt{2})
\]

**Pf**: -Suppose not, then \( u, s.p.s \in \mathbb{Z} \)
\[
\text{s.t. } \frac{a}{b} = \sqrt{2}
\]
- We may assume \( a, b \) share no common factors, since if they did
we could cancel them factors to
get \( a', b' \in \mathbb{Z} \) without common factors.
s.t. \( \frac{a'}{b'} = \sqrt{2} \).

Now: \(-\sin a \frac{a}{b} = \sqrt{2}\)

we have \( a = \sqrt{2}b \)

\( \Rightarrow a^2 = 2b^2 \)

hence \( a^2 \) is even. It follows that \( a \) is even (why?)

- hence \( \exists k \in \mathbb{Z} \) s.t. \( a = 2k \)
- then \( a^2 = 4k^2 \)
- giving \( 2b^2 = 4k^2 \)
- hence \( b^2 = 2k^2 \)
- we now see that \( b^2 \), and hence \( b \), is also even.
- so both \( a, b \) are even: hence they share a factor of 2
- a contradiction as \( a, b \) share no common factors!
- the proof follows.
Conditional Claim

General Form: \( P \Rightarrow Q \)

Three strat\( t \):

1. **Direct**: Assume \( P \)
   holds, Prove \( Q \)

2. **Contra-positive**: Prove \( \neg Q \Rightarrow \neg P \)
   i.e. assume \( \neg Q \) and prove \( \neg P \)

3. **Indirect**: Assume \( \neg (P \Rightarrow Q) \)
   (equiv: \( P \land \neg Q \)) and derive contradiction

2 and 3 often similar in practice

**Ex 1** (Direct) \( \mathbb{W} \): \( O = \{-\ldots,-5,-3,-1,1,3,5,\ldots\} \)

denote the set of all odd integers.

**Prop'n** \( (\forall n \in \mathbb{Z}) \left( n \in O \Rightarrow n^2 - 1 \text{ is divisible by } 4 \right) \)

i.e. \( (\forall n \in \mathbb{Z}) \left( n \in O \Rightarrow (\exists k \in \mathbb{Z}) \left( n^2 - 1 = 4k \right) \right) \)

**Pf**: (overall this is a universal claim, so we begin as usual)

- Fix \( n \in \mathbb{Z} \)
  (now we deal w/ the conditional)
- Assume $n \neq 0$.
  (allowed to do this since if $n \neq 0$
  the conditional claim holds vacuously)
- hence $\exists k \epsilon \mathbb{Z}$ s.t. $n = 2k+1$.
- hence $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$
  $\implies n^2 - 1 = 4k^2 + 4k$
  $= 4(k^2 + k)$
  $= 4M$ (where $M = k^2 + k$)
- hence $n^2 - 1$ is divisible by $4$.
- since $n \epsilon \mathbb{Z}$ was arbitrary, the
  claim is proved. $\checkmark$

$\text{Ex}\, \oplus (\text{contrapositive}). \ W.E = \{-\ldots, -4, -2, 0, 2, 4, \ldots\}$

$\text{Propn} \ (\forall m, n \epsilon \mathbb{Z}) \ (m \epsilon E \implies (m \epsilon E) \lor (n \epsilon E))$

$\text{Pf.} \ - \ \text{fix } m, n \epsilon \mathbb{Z}$
  (we argue the contrapositive by contrapositive)
- assume $\neg (m \epsilon E \lor n \epsilon E)$
  i.e. $m \notin E \land n \notin E$.
- then $m, n$ are both odd, i.e.
  $\exists k, l \epsilon \mathbb{Z}$ s.t. $m = 2k+1$
  $n = 2l+1$
\[ mn = (2k+1)(2l+1) \]
\[ = 4kl + 2k + 2l + 1 \]
\[ = 2(2kl + k + l) + 1 \]
\[ = 2N + 1 \quad \text{(where } N = 2kl + k + l) \]

- hence \( mn \) is odd, i.e. \( mn \notin \mathbb{E} \).

- we're proved 
  \( (m \notin \mathbb{E} \lor n \notin \mathbb{E}) \Rightarrow mn \notin \mathbb{E} \)
  i.e. \( \neg (m \in \mathbb{E} \lor n \in \mathbb{E}) \Rightarrow \neg (mn \in \mathbb{E}) \)

- by contrapositive we've proved 
  \( mn \in \mathbb{E} \Rightarrow m \in \mathbb{E} \lor n \in \mathbb{E} \)

- since \( m, n \in \mathbb{Z} \) were arbitrary, claim is proved

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**Ex 3 (Indirect)**

**Prop'n** \( \forall x \in \mathbb{R} \left( x > 0 \Rightarrow x + \frac{1}{x} \geq 2 \right) \)

**Pf:**
- fix \( x \in \mathbb{R} \)
- suppose \( x > 0 \) but \( x + \frac{1}{x} < 2 \)
- \( \neg Q \)

\[ \Rightarrow x^2 + 1 < 2x \quad \text{(inequality doesn't flip since } x > 0) \]
\[ x^2 - 2x + 1 < 0 \]
\[ \Rightarrow (x-1)^2 < 0. \]

- a contradiction, as squares are always \( \geq 0 \).

- hence we must have

\[ x > 0 \Rightarrow x + \frac{1}{x} \geq 2. \]

- since \( x \) was arbitrary, claim is proved.

**Biconditional Claims**

**General Form:** \( P \iff Q \)

**Strategy:** Prove \( P \Rightarrow Q \) and \( Q \Rightarrow P \)

**Ex:** Prep'n An integer \( n \) is even iff its square is even, i.e.

\[ (\forall n \in \mathbb{Z}) (n \in E \iff n^2 \in E) \]

**Pf:** - fix \( n \in \mathbb{Z} \)

\[ (\Rightarrow) - \text{Assume } n \in E \]

- then \( \exists k \in \mathbb{Z} \) s.t. \( n = 2k \)
- Then \( n^2 = 4k^2 \)
  \[ = 2(2k^2) \]
  \[ = 2M \quad \text{(where } M = 2k^2) \]

- Hence \( n^2 \) is even, i.e. \( n \in E. \checkmark \)

(\( \Leftarrow \)) To prove \( n^2 \in E \Rightarrow n \in E \) we show the contrapositive: \( n \notin E \Rightarrow n^2 \notin E \).

- So suppose \( n \notin E \).

- Then \( n \) is odd, i.e. \( \exists k \in \mathbb{Z} \) s.t. \( n = 2k + 1 \)

- Hence \( n^2 = (2k + 1)^2 \)
  \[ = 4k^2 + 4k + 1 \]
  \[ = 2(2k^2 + 2k) + 1 \]
  \[ = 2N + 1 \quad \text{(where } N = 2k^2 + 2k) \]

- Hence \( n^2 \) is odd, so \( n^2 \notin E \).

- We've shown (by contrapositive) \( n^2 \notin E \Rightarrow n \notin E \).

Thus: \( n \in E \Leftrightarrow n^2 \in E \)

- Since \( n \) was arbitrary, premise \( n \) was proved \( \checkmark \).