A \cap B = \{x \in U \mid (x \in A) \land (x \in B)\}
A \cup B = \{x \in U \mid (x \in A) \lor (x \in B)\}
\overline{A} = \{x \in U \mid \neg (x \in A)\}

-equiv to writing “x \notin A”

— we’ll explore connections between connectives and set operations more later.

### Implication

Given statements P, Q, the statement \( P \implies Q \) is read “if P, then Q” or “P implies Q”.

\( P \implies Q \) is true if whenever \( P \) is true, \( Q \) is also true.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P \implies Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Notice: \( \neg P \implies Q \) is always true when \( P \) is false (often a confusing point) - \( P \implies Q \) only false when \( P \) is true and \( Q \) is false.
- statements of the form \( P \Rightarrow Q \) are called “conditional statements.”

**Ex.**

1. “\((1+1=2) \Rightarrow (1+1+1=3)\)” is true
   \[
   \begin{array}{c|c}
   P & Q \\
   \hline
   T & T \\
   
   \end{array}
   \]

2. “\((1+1=2) \Rightarrow (1+1+1=4)\)” is false
   \[
   \begin{array}{c|c}
   P & Q \\
   \hline
   T & T \\
   
   \end{array}
   \]

3. “\((1+1=2) \Rightarrow (\sqrt{2} \notin N)\)” is true even though \( P, Q \) in this example are apparently unrelated statements.

4. “My name is Sally \( \Rightarrow \) My name begins with S” is true
   (both premise \( P \) and conclusion \( Q \) are false, therefore (by def'inition) \( P \Rightarrow Q \) is true ("false \Rightarrow false" is \( T \))

5. “Tomorrow \( \Rightarrow \) Sunday” \( \Rightarrow \) My name is Gorrell” is also true
   ("false \Rightarrow true" is \( T \))
6. \((\exists x \in \mathbb{R})(x^2 = -1) \implies (1 + 1 = 3)\)

is true: automatically since premise is false, though unrelated to conclusion.

7. Can also use \(\implies\) in var. propn:

- e.g. \(x \geq 2 \implies x^2 \geq 4\)

is a var. propn and

\((\forall x \in \mathbb{R})(x \geq 2 \implies x^2 \geq 4)\)

is a true statement, since

for every \(x \in \mathbb{R}\)

- either \(x \geq 2\) is true, in which case \(x^2 \geq 4\) is also true, and "true \(\implies\) true" is true

or \(x < 2\) is false, in which case \(x \geq 2 \implies x^2 \geq 4\) is true automatically.

i.e. for every \(x \in \mathbb{R}\), "\((x \geq 2) \implies (x^2 \geq 4)\)" is (T), i.e. \((\forall x \in \mathbb{R})(x \geq 2 \implies x^2 \geq 4)\) is (T).
8. \( (\forall x \in \mathbb{R}) (x^2 \geq 4 \Rightarrow x \geq 2) \) is false since there is a real number \( x \) (e.g. \( x = -3 \)) s.t. \( x^2 \geq 4 \) but \( x \geq 2 \) is false.

**Equivalence**

Given statements \( P, Q \), the statement \( P \iff Q \) (read: "\( P \) if and only if \( Q \)")

is true if \( P \), \( Q \) have the same truth values.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>( P \iff Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

**Examples**

1. \( (1 + 1 = 2) \iff (1 + 1 = 3) \) is (F)
2. \( (1 + 1 = 3) \iff (1 + 1 = 4) \) is (T)
3. \( (\forall x \in \mathbb{N})(n > 0) \iff (1 + 1 = 2) \) is (T)
4. \( (1 + 1 = 2) \iff (2 + 2 = 5) \) is (F)

Can also use \( \equiv \) in var prop'n, e.g.

\[ (x > 0) \equiv (\exists y \in \mathbb{R})(y^2 = x) \]

is a legit var. prop'n
and \((\forall x \in \mathbb{R})[(x > 0) \iff (\exists y \in \mathbb{R})(y^2 = x)]\) is a true statement, since:

for every \(x \in \mathbb{R}\) the statements 
\(x > 0\) and 
\((\exists y \in \mathbb{R})(y^2 = x)\) are either both true or both false.

Defin statements \(P, Q\) are said to be logically equivalent if they have the same truth value, i.e.

\[\text{iff } P \iff Q \text{ is true.}\]

- e.g. \(1 + 1 = 2\) and \(1 + 1 + 1 = 3\) are logically equiv.

- will be most interested in logically equivalent forms for
connected (esp. negated) and
quantified statements.

Negating Quantified statements

- SPS \(P(x)\) is a variable prop'n
and \(S\) is a set.
Consider the negated statements:

1. \( \neg (\forall x \in S) P(x) \)
2. \( \neg (\exists x \in S) P(x) \)

Observe:
1. is true iff there is no \( x \in S \) s.t. \( P(x) \) is false, i.e. iff 
   \( (\exists x \in S) \neg P(x) \) is true
2. is true iff for all \( x \in S \) we have \( P(x) \) is false, i.e.
   \( (\forall x \in S) \neg P(x) \) is true.

This shows:
\[ \neg (\forall x \in S) P(x) \iff (\exists x \in S) \neg P(x) \]

is always true (regardless of \( P(x) \)), i.e. that \( \neg (\forall x \in S) P(x) \) and \( (\exists x \in S) \neg P(x) \) are logically equivalent.

Likewise,
\[ \neg (\exists x \in S) P(x) \text{ and } (\forall x \in S) \neg P(x) \]
are logically equivalent.
these equivalences often useful
when trying to prove quantified
statements by contradiction.

\[ \neg (\forall x \in \mathbb{R}) (x \in \mathbb{N}) \]

is equiv. to

\[ (\exists x \in \mathbb{R}) \neg (x \in \mathbb{N}) \]

both are (T)

\[ \text{note: we'll often write } \neg (x \in \mathbb{N}) \]
\[ \text{as } x \notin \mathbb{N}, \quad \neg (x = y) \Rightarrow x \neq y, \text{ etc.} \]

2 \[ \neg (\exists x \in \mathbb{R}) (x + 1 = 0) \]
is equiv. to

\[ (\forall x \in \mathbb{R}) (x + 1 \neq 0) \]

(both (F))

3 For multiple quantifiers, iterate
the process:

\[ \neg (\forall x \in \mathbb{R})(\exists y \in \mathbb{R}) (xy = 1) \rightarrow \text{ "not every real has a mult. imurr"} \]

equiv. to:

\[ (\exists x \in \mathbb{R}) \neg (\exists y \in \mathbb{R}) (xy = 1) \]

also:

\[ (\exists x \in \mathbb{R})(\forall y \in \mathbb{R}) (xy \neq 1) \]
(both are (T):
O has no multiplicative inverse)

"there is a real w/o a multiplicative inverse"