(i)

Nonexamples:

① $\leq$ is not a strict partial order on $\mathbb{R}$ since $\leq$ is not irreflexive (in fact, $\leq$ is reflexive which is stronger than being not irreflexive). Sim'ly, $\subseteq$ is not a strict p.o.

② OTOH, $<$ and $\subsetneq$ are not (nonstrict) partial orders: neither are reflexive (in fact: irreflexive).

③ More generally, a relation R cannot be both a strict and non-strict p.o. (unless $R = \emptyset$ is the empty order): can't be both reflexive and irreflexive

④ $\neq$ (e.g. on $\mathbb{N}$) is neither a strict nor nonstrict p.o. since $\neq$ is not transitive.

Total orders:

Def'n: a relation R on A is said to be total iff

$$(\forall x, y \in A)\big((x,y) \in R \vee (y,x) \in R \vee x = y\big)$$

Def'n: - if R is a partial order on A that is also total, then R is called a total order on A.

(ii)

- If R is a strict partial order on A that is also total, then R is called a strict total order on A.

Ex's ① $\leq$ is a total order on $\mathbb{R}$: we know $\leq$ is a p.o. and further:

$$(\forall x,y \in \mathbb{R})\ (x \leq y \vee y \leq x \vee x = y) \quad \checkmark$$

② $\subseteq$ is not a total order on $P(\mathbb{N})$: it is not total, e.g.:

If $X = \{1,2\}$ $Y = \{2,3\}$

then $X \not\subseteq Y$, $Y \not\subseteq X$, $X \neq Y$.

③ Simly, $|$ is a p.o. on $\mathbb{N}$ but is not total, e.g.

$$3 \nmid 5, \quad 5 \nmid 3, \quad 3 \neq 5.$$

④ $<$ is a strict total order on $\mathbb{R}$: we know it's a strict p.o. and it's total since:

$$(\forall x,y \in \mathbb{R})\ (x < y \vee y < x \vee x = y).$$

⑤ $\subsetneq$ is not a strict total order on $P(\mathbb{N})$: it's not total.

(iii)

Functions

- Functions, like relations, are ubiquitous in math. But what "are" functions?
- Intuitively: a function is a rule that assigns to each input x in a domain A a unique output $f(x)$ in a codomain B.

Idea: can define functions rigorously as a special type of relation.

Def'n: a function (with domain A and codomain B) is a relation $f \subseteq A \times B$ s.t.

for every $a \in A$
there is a unique $b \in B$
s.t. $(a,b) \in f$.

i.e.

$$(\forall a \in A)(\exists b \in B)\left[(a,b) \in f \wedge (\forall c \in B)\left((a,c) \in f \Rightarrow c = b\right)\right].$$

Notation: we write

$$f : A \to B$$

to mean a relation $f \subseteq A \times B$ is a function.

(iv) We also write:

$$f(a) = b$$

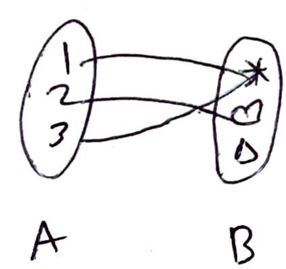
to mean $(a,b) \in f$.

Note: - defn says: every $a \in A$ assigned an output $f(a) \in B$.

- does <u>not</u> insist for every $b \in B$ there is $a \in A$ s.t. $f(a) = b$ (functions w/ this property are called surjective).

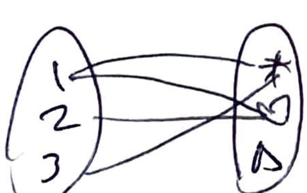
Ex's: ① Let $A = \{1,2,3\}$, $B = \{*, \heartsuit, \Delta\}$.

Then: $f = \{(1,*), (2,\heartsuit), (3,*)\}$ is a function from A to B:



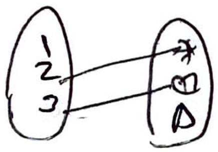
a function

<u>but</u> $g = \{(1,*), (1,\heartsuit), (2,\heartsuit), (3,*)\}$ is <u>not</u> since 1 does not have a unique output: both $(1,*)$ and $(1,\heartsuit) \in g$.



<u>not</u> a function.

(v) not as $h = \{(2, *), (3, \heartsuit)\}$ since 1 is not assigned an output.



not a function

② We'll often define functions by a <u>rule</u>:

e.g. $f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

or: $g : \mathbb{R} \to \mathbb{Z}$

$g(x) = \lfloor x \rfloor$.

but behind the scenes these f's are still <u>sets</u> <u>of ordered</u> <u>pairs</u>:

e.g. for f above we have:

$(2, 4) \in f$

$(3, 9) \in f$

but $(7, 12) \notin f$ (since $f(7) = 49 \neq 12$)

<u>Warning</u>: not all rules yield well-defined functions!

e.g. suppose we "define"

$f : \mathbb{Q} \to \mathbb{Z}$

by the rule $f(m/n) = m + n$.

(vi)

then this "function" is not one:

$$f(1/2) = 1+2 = 3 \neq 6 = 2+4 = f(2/4)$$

but $1/2 = 2/4$.

- so f assigns multiple outputs to the same input. what's going on?
- Really: there's an implicit equiv. relation defined on fraction representations $(\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots)$ and our rule for f is defined in terms of the representative of an equiv. class, not the class itself.

- in general: when given a rule "defining" $f \subseteq A \times B$, to verify f is a function we must show: ① $(\forall a \in A)(\exists b \in B)\ (a,b) \in f$

② if $a = a'$ then $f(a) = f(a')$.

Equality of functions.

Q: What does it mean for functions $f: A \to B$ and $g: A \to B$ to be equal?

A: $f = g$ iff they're equal as sets (of ordered pairs) i.e. iff $f \subseteq g$ and $g \subseteq f$.

(vii)

In practice: often ez'er to use following criterion:

Thm: if $f: A \to B$ and $g: A \to B$ are functions then $f = g$ iff

$$(\forall a \in A)(f(a) = g(a))$$

Pf: you try.

The point: functions can be equal despite being defined by different rules.

e.g.: let $A = \{1, 2, 3\}$

define $f: A \to \mathbb{N}$

$g: A \to \mathbb{N}$

by $f(x) = x^3 + 11x$

$g(x) = 6x^2 + 6$

then:

$$f(1) = 12 = g(1)$$
$$f(2) = 30 = g(2)$$
$$f(3) = 60 = g(3)$$

i.e. $f = \{(1,12), (2,30), (3,60)\} = g$.

(What's the magic trick?

$f - g = x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$

$= 0$ for $x \in \{1,2,3\}$)

(viii)

Images

Def'n: Sps $f: A \to B$ is a function and $X \subseteq A$.
The image of X under f, denoted $Im_f(X)$,
is defined as:

$$Im_f(X) = \{b \in B \mid (\exists a \in X)\ f(a) = b\}$$

informally we could write:

$$= \{f(a) \mid a \in A\}$$

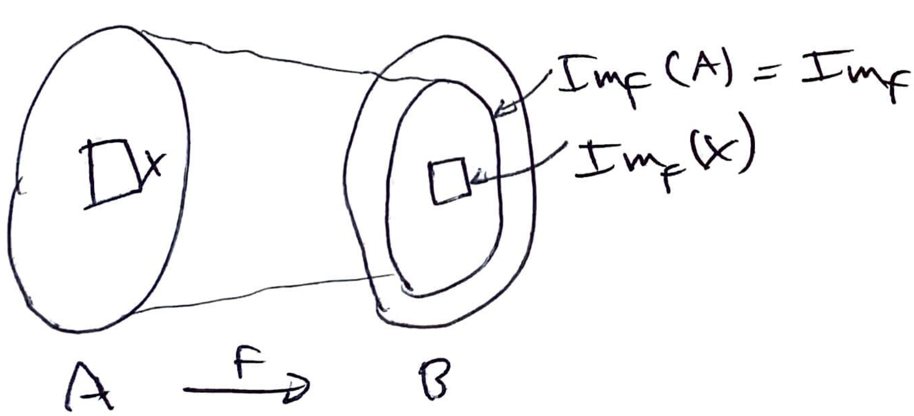
the point: if $a \in A$ then $f(a) \in Im_f(A)$.

↳ when $X = A$ we just say $Im_f(A)$ is the image of f and sometimes just write Im_f.

Def'n says: – $Im_f(X)$ is the "set of outputs of el'ts in X"

– $Im_f = Im_f(A)$ is the "set of all outputs".

Picture:



(ix) Ex's ① Let $A = \{1,2,3\}$ $B = \{*, \heartsuit, \Delta\}$

$f : A \to B$ be $\{(1,*), (2,\heartsuit), (3,*)\}$

Then: - $\mathrm{Im}_f(\{1,3\}) = \{f(1), f(3)\}$

$= \{*, *\}$

$= \{*\}$.

- $\mathrm{Im}_f = \mathrm{Im}_f(\{1,2,3\}) = \{f(1), f(2), f(3)\}$

$= \{*, \heartsuit, *\}$

$= \{*, \heartsuit\}$.

② Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

Then: - $\mathrm{Im}_f(\{-1, 0, 1\}) = \{(-1)^2, 0^2, 1^2\}$

$= \{0, 1\}$.

- $\mathrm{Im}_f = \{x \in \mathbb{R} \mid x \geq 0\}$.

Functions add a layer of complexity to the basic set theory of $\cup, \cap, \ldots$ that we studied earlier. E.g.:

Prop'n: Sps $f : A \to B$ is a function and $S, T \subseteq A$.

Then: $\mathrm{Im}_f(S \cap T) \subseteq \mathrm{Im}_f(S) \cap \mathrm{Im}_f(T)$.

(x)

Pf: - fix $y \in Im_f(S \cap T)$

- then $\exists x \in S \cap T$ s.t. $f(x) = y$
- hence $x \in S$ and $x \in T$
- hence $f(x) \in Im_f(S)$ and $f(x) \in Im_f(T)$
- i.e. $y \in Im_f(S)$ and $y \in Im_f(T)$
- i.e. $y \in Im_f(S) \cap Im_f(T)$

Since y was arbitrary the prop'n is proved.

Note: in general we don't have

$$Im_f(S \cap T) = Im_f(S) \cap Im_f(T)$$

e.g. Consider $f(x) = x^2$ on $\mathbb{R}$.

Let $S = \{-1, 0\}$ $T = \{0, 1, 2\}$

then $Im_f(S) = \{f(-1), f(0)\} = \{1, 0\}$

$Im_f(T) = \{f(0), f(1), f(2)\} = \{0, 1, 4\}$

$\Rightarrow Im_f(S) \cap Im_f(T) = \{0, 1\}$

but: $Im_f(S \cap T) = Im_f(\{0\})$

$= \{f(0)\} = \{0\}$.

So in this case: $Im_f(S \cap T) \subsetneq Im_f(S) \cap Im_f(T)$.

the essence of the issue: functions can send multiple inputs to the same output.