Notice: in both examples ① and ② the set of equiv. classes forms a partition of the underlying set ($\mathbb{Z}$ in ①, $\mathbb{R}$ in ②)

- turns out this is always the case:

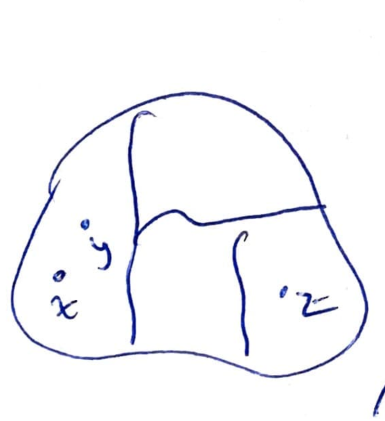
Theorem: If R is an equiv. relation on A, then A/R is a partition of A.

Pf: HW. For a hint, see 6.7.13 on pg. 449, which outlines an approach to the proof.

Partitions yield Equiv. relations

idea: if $\mathbb{P}$ is a partition on A, can define an equiv. relation R on A by rule "$(x,y) \in R$ iff x and y are in same piece of partition"

Picture:



$(x,y) \in R$
but $(x,z) \notin R$.

Let's prove this works:

Theorem Sps $\mathbb{P}$ is a partition of A. Define a relation $R_{\mathbb{P}}$ on A by:

$$(x,y)\in R_{\mathbb{P}} \text{ iff } (\exists X\in \mathbb{P})(x\in X \text{ and } y\in X)$$

Then: $R_{\mathbb{P}}$ is an equiv. relation.

Pf: (i) (reflexivity)

- Fix $x\in A$
- Since $\mathbb{P}$ is a partition we know

$$\bigcup_{X\in\mathbb{P}} X = A$$

- So since $x\in A = \bigcup_{X\in\mathbb{P}} X$ there is $X\in\mathbb{P}$ s.t. $x\in X$
- hence $x\in X$ also
- hence $(x,x)\in R_{\mathbb{P}}$ ✓

(ii) (symmetry)

- Fix $x,y\in A$ and sps $(x,y)\in R_{\mathbb{P}}$
- by def'n of $R_{\mathbb{P}}$, there is some $X\in\mathbb{P}$ s.t. $x\in X$ and $y\in X$.
- hence $y\in X$ and $x\in X$
- hence $(y,x)\in R_{\mathbb{P}}$. ✓

(iii) (transitivity)

- Fix $x, y, z \in A$ and sps $(x,y) \in R_{\mathbb{P}}$ and $(y,z) \in R_{\mathbb{P}}$.
- then by def'n of $R_{\mathbb{P}}$, there is some $X \in \mathbb{P}$ s.t. $x \in X$ and $y \in X$
- also there is $Y \in \mathbb{P}$ s.t. $y \in Y$ and $z \in Y$.
- hence $y \in X \cap Y$
- in particular $X \cap Y \neq \emptyset$.
- but then $X = Y$, since $\mathbb{P}$ is a partition
- hence $x \in X$ and $z \in \overset{=Y}{X}$
- hence $(x,z) \in R_{\mathbb{P}}$ ✓.

<u>Ex's</u>: ① - Let $\mathbb{P} = \{X, Y, Z\}$

where $X = \{\ldots, -3, 0, 3, 6, \ldots\}$

$Y = \{\ldots, -2, 1, 4, 7, \ldots\}$

$Z = \{\ldots, -1, 2, 5, 8, \ldots\}$

be our partition of $\mathbb{Z}$ from before

- Let $R_{\mathbb{P}}$ be the associated equiv. relation:

$(x,y) \in R_{\mathbb{P}}$ iff $\exists S \in \mathbb{P}$ s.t. $x \in S$ and $y \in S$.

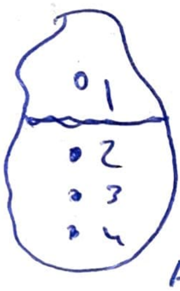
- by our theorem, this defines an equivalence relation.

- easy to see this is the same equiv. relation $\equiv_3$ that we defined previously in a different way:

$$x \equiv_3 y \quad \text{iff} \quad 3 \mid y - x.$$

- notice: the equiv. classes of the relation are exactly the pieces of the partition.

② - let $\mathbb{P} = \{\{1\}, \{2,3,4\}\}$.
  - Then $\mathbb{P}$ is a partition of the set $A = \{1,2,3,4\}$ into 2 pieces.
  - let $R_{\mathbb{P}}$ be the associated equiv. relation
    ↳ so e.g. $(1,1) \in R_{\mathbb{P}}$
    $(2,4) \in R_{\mathbb{P}}$
    but $(1,3) \notin R_{\mathbb{P}}$



A

- In this case we can actually <u>write out</u> $R_{\mathbb{P}}$ as a set of ordered pairs, explicitly, in roster notation.

$$R_{\mathbb{P}} = \{(1,1), (2,2), (3,3), (4,4), (2,3), (3,2), (2,4), (4,2), (3,4), (4,3)\}.$$

- no real rhyme or reason to this equiv. relation, but still a perfectly good one.

Order Relations

- another common type of binary relation is an order relation.
- unlike equiv. relations, order relations come in several flavors:
  - nonstrict / strict
  - partial / total.

Def'n A relation R on a set A is a (nonstrict) partial order if R is reflexive, transitive, and antisymmetric.

↳ if R is a partial order on A we say that the pair (A, R) is a partially ordered set, or poset.

Ex's ① $\leq$ is a partial order on $\mathbb{R}$.

why: $\forall x, y, z \in \mathbb{R}$ we have:

(i) $x \leq x$ ✓
(ii) if $x \leq y$ and $y \leq z$ then $x \leq z$ ✓
(iii) if $x \leq y$ and $y \leq x$ then $x = y$ ✓

↳ so $(\mathbb{R}, \leq)$ is a poset.

② Let A be a fixed set. Then the subset relation $\subseteq$ on $P(A)$ is a partial order

Why: $\forall X, Y, Z \in P(A)$ we have:

(i) $X \subseteq X$ ✓

(ii) if $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$ ✓

(iii) if $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$ ✓

↳ so $(P(A), \subseteq)$ is a partially ordered set

③ We showed before that the divisibility relation $|$ on $\mathbb{N}$ (i.e. $n | m$ iff $\exists k \in \mathbb{N}$ $m = nk$) is reflexive, transitive, antisymmetric

hence $(\mathbb{N}, |)$ is a poset.

Question is $(\mathbb{Z}, |)$ a poset?

we still have reflexivity and transitivity

What about antisymmetry?

if $n|m$ and $m|n$ do we have $n = m$, if $n, m \in \mathbb{Z}$?

no! Consider 2 and -2

$2 | -2$ and $-2 | 2$ but $-2 \neq 2$.

So the divisibility relation $|$ on $\mathbb{Z}$ is not antisymmetric, hence $(\mathbb{Z}, |)$ is not a poset.

↳ these examples of partial orders seem to be of different kinds — and yet: any theorems that can be proved about them using only the properties of reflexivity, transitivity and antisymmetry must be true for all ~~3~~ three! (and any other poset).

Strict partial orders

Def'n: a relation R on A is irreflexive iff $(\forall x \in A)\ (x,x) \notin R$.

e.g. $<$ and $\neq$ are irreflexive since we never have $x<x$ or $x \neq x$.

Def'n a relation R on a set A is called a strict partial order iff R is (i) irreflexive (ii) transitive (iii) antisymmetric

↳ this is official def'n, but by HW this is same as saying: R is asymmetric and transitive i.e. $(\forall x, y \in A)\ (x,y) \in R \Rightarrow (y,x) \notin R$.

Ex's ① $<$ is a strict partial order on $\mathbb{R}$.

Pf: $\forall x, y, z \in \mathbb{R}$ we have:

(i) $x \not< x$ ✓

(ii) $x < y$ and $y < z \Rightarrow x < z$ ✓

(iii) if $x < y$ and $y < x$ ~~then~~ $\Rightarrow y = x$ ✓

because always false.

by HW: instead of checking (i) and (iii) can instead observe:

(iv) $x < y \Rightarrow y \not< x$.

② Let A be a fixed set. Then $\subsetneq$ is a strict p.o. on $P(A)$.

Pf: $\forall X, Y, Z \in P(A)$

(i) $X \subsetneq Y$ and $Y \subsetneq Z \Rightarrow X \subsetneq Z$ ✓

(ii) if $X \subsetneq Y$ then Y is not $\subsetneq X$. ✓